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Algorithms for Computation of Water Level Elevation at a Fixed Location From the Water Level Elevations at a Moving Platform

ABSTRACT Kriging and conditional simulation algorithms have been developed for estimating the water level elevation at a fixed reference location from the measured water level locations on a moving platform. Theory and mathematical procedures are presented.

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The algorithms for the kriging and conditional simulation are quite straightforward once the covariance matrix is developed. The main task to be completed is the development of a rapid procedure for computing the covariances. The major part of the report will be concerned with this problem.

THE SPATIAL–TEMPORAL COVARIANCE FUNCTION

Let (x_1, y_1) and (x_2, y_2) be any two locations and (t_1, t_2) be any two times. The water level elevation above mean water level at (x, y, t) will be denoted by $\eta(x, y, t)$.

The covariance between two random variables U and V is defined as the statistical average of $(U - \mu_u)(V - \mu_v)$ where μ_u is the statistical average of U and μ_v is the statistical average of V. It is convenient to denote the statistical averaging process with E [ ]. Then the covariance definition may be expressed in symbols as

$$\mathrm{Cov}(U, V) = E[(U - \mu_u)(V - \mu_v)]$$

where

$$E[\,U\,] = \mu_u$$
$$E[\,V\,] = \mu_v$$

Now replacing U with $\eta(x_1, y_1, t_1)$ and V with $\eta(x_2, y_2, t_2)$, and noting that, by definition of η, the average of $\eta(x_1, y_1, t_1)$ and $\eta(x_2, y_2, t_2)$ is zero, the covariance between the two can be written as

$$\mathrm{Cov}[\eta(x_1, y_1. t_1), \eta(x_2, y_2, t_2)] = E[\eta(x_1, y_1, t_1)\ \eta(x_2, y_2, t_2)]$$

That is, the covariance is the expected product.

It will be asusmed that the sea surface is a stationary, Gaussian process. Then the covariance defined above will only be a function of the differences in position and time. Let

$$\tau = t_2 - t_1$$

$$X = x_2 - x_1$$

$$Y = y_2 - y_1$$

be these differences, and define the covariance function as $C(X, Y, \tau)$ where

$$C(X, Y, \tau) = \mathrm{Cov}[\eta(x_1, y_1, t_1), \eta(x_2, y_2, t_2)]$$

It can be shown that (Borgman, 1969, p. 723) the spatial, temporal covariance function can be expressed in terms of the directional spectrum as

$$C(X, Y, \tau) = 2\int_0^{\infty}\int_0^{2\pi} S(f,\theta)\cos[wk(X\cos\theta + Y\sin\theta) - 2\pi f\tau]\,d\theta\,df$$

where

$$w = \begin{cases} +1, \text{ if } \theta \text{ is the direction toward which waves travel} \\ -1, \text{ if } \theta \text{ is the direction from which waves travel} \end{cases}$$

k = wave number

= function of f defined by $(2\pi f)^2 = gk\tanh(kd)$

d = water depth

g = acceleration due to gravity.

The functions involved can be defined for negative frequency by

$$S(-f, \theta) = S(f, \theta)$$

$$k(-f) = -k(f)$$

Then, through the complex–valued definition of the cosine as

$$\cos \phi = [\exp(i\ \phi) - \exp(-i\ \phi)]/2$$

the formula for $C(X,\ Y,\ \tau)$ can be rewritten as

$$C(X,\ Y,\ \tau) = \int_{-\infty}^{\infty} \int_{0}^{2\pi} S(f,\ \theta)\ \exp[-iwk(X\cos\theta + Y\sin\theta) + i2\pi f\tau] d\theta df$$

A POLAR FORM FOR THE COVARIANCE FUNCTION

Suppose (X, Y) is re–expressed in polar coordinates (ρ, α), with

$$X = \rho \cos \alpha$$

$$Y = \rho \sin \alpha$$

then

$$X \cos \theta + Y \sin \theta = \rho \cos \alpha \cos \theta + \rho \sin \alpha \sin \theta = \rho \cos(\theta - \alpha)$$

With these definitions

$$C(X, Y, \tau) = \int_{-\infty}^{\infty} \int_{0}^{2\pi} S(f, \theta) \exp[-iwk\, \rho \cos(\theta - \alpha)] \exp[i2\pi f \tau] d\theta df$$

Without loss of generality, $S(f, \theta)$ can be expressed in the product form

$$S(f, \theta) = S(f)\, D_f(\theta)$$

where $D_f(\theta)$ is the spreading function defined so that

$$\int_{0}^{2\pi} D_f(\theta) d\theta = 1.0$$

$$D_f(\theta) \geq 0$$

The spreading function gives the distribution of wave energy or variance with direction at frequency f. The covariance function can be written in terms of the product form as

$$C(X, Y, \tau) = \int_{-\infty}^{\infty} S(f) \left\{ \int_{0}^{2\pi} D_f(\theta)\, e^{-iwk\rho \cos(\theta-\alpha)} d\theta \right\}$$

$$* \, e^{i2\pi f\tau} df = C^*(\rho, \alpha)$$

(where * denotes multiplication).

In summary, the covariance can be expressed in rectangular and polar form as

$$C(X, Y, \tau) = 2\int_{0}^{\infty} S(f) \left\{ \int_{0}^{2\pi} D_f(\theta) \cos[wk(X\cos\theta + Y\sin\theta) - 2\pi f\tau] d\theta \right\} df$$

$$C^*(\rho, \alpha, \tau) = \int_{-\infty}^{\infty} S(f) \left\{ \int_{0}^{2\pi} D_f(\theta)\, e^{-iwk\rho\cos(\theta-\alpha)} d\theta \right\} e^{i2\pi f\tau} df$$

Either form can be used as a basis for computaitons. In practice, it is usually best to pre compute a table of the Covariance function values for a given $S(f, \theta)$ and a grid of values,

$$-h_0 \leq X \leq h_0$$
$$-h_0 \leq Y \leq h_0$$
$$0 \leq \tau \leq \tau_0$$

or

$$0 \leq \rho \leq \rho_0$$
$$0 \leq \alpha \leq 2\pi$$
$$0 \leq \tau \leq \tau_0$$

It is only necessary to compute these for positive time lag because, from the definitions

$$C(-X, -Y, -\tau) = C(X, Y, \tau)$$

$$C^*(\rho, \alpha+\pi, -\tau) = C^*(\rho, \alpha, \tau)$$

For the Navy application, where the $S(f,\theta)$ function is estimated from a buoy, further simplications are often appropriate. It is often satisfactory to take $D_f(\theta)$ as only depending on θ. When this happens, the spreading function will be denoted by $D(\theta)$. Three common formulas for the spreading function that are used as approximations are

The generalized cosine–squared model,

$$D(\theta) = c\cos^{2s}[(\theta-\mu)/2]$$

The von Mises model,

$$D(\theta) = \{\exp[a\cos(\theta-\mu)]\}/\{2\pi I_0(a)\}$$

where $I_0(a)$ = modified Bessel function of order zero.

and the wrapped–normal model

$$D(\theta) = \sum_{j=-\infty}^{\infty} \exp\left[-\left[\frac{\theta-\mu+2\pi j}{\sigma}\right]^2/2\right]/(\sqrt{2\pi}\,\sigma)$$

All models have about the same shape and are unimodal and symmetric about μ. Approximate equivalent values between s, a, and σ^2 are given in the appendix to the report.

Since all three models have very nearly the same shape for most typical wave spreading conditions, it is just a matter of mathematical convenience which is used. If one is a reasonable approximation, then any other of the three will also be a reasonable approximation.

In the polar covariance function with the spreading function independent of frequency, it is particularly convenient to use the von Mises model.

$$C^*(\rho,\alpha,\tau) = \int_{-\infty}^{\infty} S(f)\left\{ \int_0^{2\pi} \frac{e^{a\cos(\theta-\mu)\ -\ iwk\rho\cos(\theta-\alpha)}d\theta}{2\pi I_0(a)} \right\} e^{i2\pi f\tau}df$$

A BESSEL FUNCTION SERIES REPRESENTATION

There is a nice series approximation of

$$e^{a\cos(\theta-\mu)}$$

in terms of the modified Bessel functions given by Oliver (1964, p. 376, eq. 9.6.34) as

$$e^{a\cos(\theta-\mu)} = I_0(a) + 2\sum_{n=1}^{\infty} I_n(a)\cos[n(\theta-\mu)]$$

After a little algebra, the formula for C* can be expressed in series form as

$$C^*(\rho,\ \alpha,\ \tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(f)\int_0^{2\pi} e^{-iwk\rho\cos(\theta-\alpha)}d\theta\, e^{i2\pi f\tau}df$$

$$+\sum_{n=1}^{\infty}\frac{I_n(a)}{\pi I_0(a)}\int_{-\infty}^{\infty} S(f)\int_0^{2\pi}\cos[n(\theta-\mu)]e^{-iwk\rho\cos(\theta-\alpha)}d\theta\, e^{i2\pi f\tau}df$$

The integration over $(0,\ 2\pi)$ is really just an integration over the full circle of 360^0. The full circle integration could just as well range over $(\alpha-\pi,\ \alpha+\pi)$. If the limits of integration are changed to this new choice and the variable of integration is changed to

$$\psi = \theta - \alpha$$

the formula simplifies to

$$C^*(\rho,\ \alpha,\ \tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(f)\int_{-\pi}^{\pi} e^{-iwk\rho\cos\psi}d\psi\, e^{i2\pi f\tau}df$$

$$+\sum_{n=1}^{\infty}\frac{I_n(a)}{\pi I_0(a)}\int_{-\infty}^{\infty}S(f)\int_{-\pi}^{\pi}\cos[n(\psi-\mu+\alpha)]e^{-iwk\rho\cos\psi}d\psi\, e^{i2\pi f\tau}df$$

The cosine in the second expression can be expanded to

$$\cos[n(\psi-\mu+\alpha)] = \cos(n\psi)\cos[n(\alpha-\mu)] - \sin(n\psi)\sin[n(\alpha-\mu)]$$

With this,

$$C^*(\rho,\ \alpha,\ \tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}S(f)\int_{-\pi}^{\pi}e^{-iwk\rho\cos\psi}d\psi\, e^{i2\pi f\tau}df$$

$$+\sum_{n=1}^{\infty}\frac{I_n(a)}{\pi I_0(a)}\cos[n(\alpha-\mu)]\int_{-\infty}^{\infty}S(f)\int_{-\pi}^{\pi}\cos(n\psi)$$

$$*\ e^{-iwk\rho\cos\psi}d\psi\, e^{i2\pi f\tau}df - \sum_{n=1}^{\infty}\frac{I_n(a)}{\pi I_0(a)}\sin[n(\alpha-\mu)]\int_{-\infty}^{\infty}S(f)\int_{-\pi}^{\pi}\sin(n\psi)$$

$$*\ e^{-iwk\rho\cos\psi}d\psi\, e^{i2\pi f\tau}df$$

The last expression is zero since the integrand

$$\sin(n\psi)\, e^{-iwk\rho\cos\psi}$$

is an odd function of ψ integrated over $(-\pi,\ \pi)$. The other two integrands

$$e^{-iwk\rho\cos\psi}$$

and

$$\cos(n\psi)\, e^{-iwk\rho\cos\psi}$$

are even functions of ψ, so the range of integration can be changed to $(0, \pi)$ with a multiplication by 2. Thus

$$C^*(\rho, \alpha, \tau) = \frac{1}{\pi}\int_{-\infty}^{\infty} S(f) \int_{0}^{\pi} e^{-iwk\rho\cos\psi} d\psi\, e^{i2\pi f\tau} df$$

$$+ 2 \sum_{n=1}^{\infty} \frac{I_n(a)}{\pi I_0(a)} \cos[n(\alpha-\mu)] \int_{-\infty}^{\infty} S(f) \int_{0}^{\pi} \cos(n\psi)$$

$$* \, e^{-iwk\rho\cos\psi} d\psi\, e^{i2\pi f\tau} df$$

The reason for all these manipulations will now become apparent. the integrals over ψ can be expressed as Bessel functions. Oliver (op.cit., p. 360, eq. 9.1.21) gives

$$\int_{0}^{\pi} e^{-iwk\rho\cos\psi} \cos(n\psi) d\psi = i^n \pi J_n(-wk\rho)$$

$$\int_{0}^{\pi} e^{-iwk\rho\cos\psi} d\psi = \pi J_0(-wk\rho)$$

By Oliver (op.cit., P. 360, eq. 9.1.20) if the argument, z, is real–valued

$$J_n(-z) = (-1)^n J_n(z)$$

Combining all these results

$$C^*(\rho,\ \alpha,\ \tau) = \int_{-\infty}^{\infty} S(f)\ J_0(k\rho)\ e^{i2\pi f\tau} df + 2 \sum_{n=1}^{\infty} (-wi)^n \frac{I_n(a)}{I_0(a)} \cos[n(\alpha-\mu)]$$

$$* \int_{-\infty}^{\infty} S(f)\ J_n(k\rho)\ e^{i2\pi f\tau}\ df$$

COMPUTATION OF THE BESSEL SERIES FORM

The last equation is in a form appropriate for approximation with the fast Fourier Transform. Let N and Δf be chosen so that $S(f)\ J_n(k\rho)$ is essentially zero for $f > N\Delta f/2$ and $n = 0, 1, 2, 3, 4, \cdots$. This is equivalent to choosing a frequency beyond which S(f) will be treated as though it is exactly zero. Then define

$$\Delta\tau = (N\Delta f)^{-1}$$

$$A_m^{(n)} = S(m\Delta f)\ J_n(k_m\ \rho)$$

where k_m is the wave number corresponding to $f = m\Delta f$, and set

$$A_{N-m}^{(n)} = \overline{A_m^{(n)}}$$

Then, introducing a new function, $R_n(\rho, \tau)$

$$R_n(\rho, j\Delta\tau) = \int_{-\infty}^{\infty} S(f)\ J_\nu(k\rho)e^{i2\pi f(j\Delta\tau)}df \approx \Delta f \sum_{m=0}^{N-1} A_m^{(n)}e^{i2\pi jm/N}$$

the function $R_n(\rho, \tau)$ can be computed quite rapidly for a selected list of ρ–values to develop a matrix whose rows are the ρ–values and whose columns are the $\tau = j\Delta\tau$ *time–lag* values.

An algorithm for the rapid computation of $J_n(z)$ for real–valued z is given by Oliver (op.cit., bottom of page 385). An exactly parallel procedure can be applied to compute $I_n(a)$ with eq. 9.6.36 (Oliver, op.cit.)

In terms of the function $R_n(\rho, \tau)$ defined above

$$C^*(\rho,\ \alpha,\ \tau) = R_0(\rho,\ \tau) + 2 \sum_{n=1}^{\infty} (-wi)^n \frac{I_n(a) R_n(\rho, \tau)}{I_0(a)} \cos[n(\alpha - \mu)]$$

The nature of values of $R_n(\rho,\ \tau)$ can be combined with any α to compute rapidly the value of $C^*(\rho,\ \alpha,\ \tau)$.

THE COVARIANCE MATRIX

Let C_{11} be the M by M matrix whose (n_1, n_2) element is the covariance between η_{n_1} and η_{n_2} for the water level elevations in the series

$$\{\eta_n,\ n = 1, 2, \cdots, M\}$$

defined in the introduction. Similarly let C_{22} be the $(M_2 - M_1+1)$ by $(M_2 - M_1+1)$ matrix whose (j_1, j_2) element is the covariance between $\eta_0(t_{M_1+j_1-1})$ and $\eta_0(t_{M_1+j_2-1})$, in the series $\{\eta_0(t_n),\ n = M_1, M_1+1, \cdots, M_2\}$ defined in the introduction. Finally let C_{12} be the M by $(M_2 - M_1+1)$ matrix whose (n, j) element is the covariance between η_n and $\eta_0(t_{M_1+j-1})$. With these definitions, the covaraince matrix of $(\underline{\eta}, \underline{\eta}_0)^T$ is

$$\mathrm{Cov}\begin{bmatrix} \underline{\eta} \\ \underline{\eta}_o \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{12}^T & C_{22} \end{bmatrix}$$

All of the covariances in C_{11}, C_{12}, and C_{22} can be computed by first determining X, Y, and τ for the pair of locations and times involved, then getting ρ and α from

$$\rho = \sqrt{X^2 + Y^2}$$

$$\alpha = \arctan\,(Y/X),$$

and finally interpolating for $R_n(\rho, \tau)$ and computing $C^*(\rho, \alpha, \tau)$.

CONDITIONAL SIMULATION OF $\underline{\eta}_o$

The matrix formula for conditional simulation of $\underline{\eta}_o$, given the values of $\underline{\eta}$, was derived previously by Borgman (1984, p. 533, eq. 85.). The following steps are involved.

(1) Compute the eigenvectors and eigenvalues of

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{bmatrix}$$

Let λ_j be the eigenvalues and $\underline{v}_j$ be the corresponding eigenvectors. Define V as a matrix whose columns are the eigenvectors and L as the diagonal matrix whose main diagonal elements are λ_j and whose off–diagonal elements are zero. Then

$$C = VLV^T$$

in the well known eigenvector, eigenvalue decomposition (Jennings, 1977, p. 32, eq. 1.130). Consequently,

$$C = (VL^{1/2})(VL^{1/2})^T$$

(2) Let $\underline{Z}^*$ be a vector of independent, standard normal random numbers. Then

$$\begin{bmatrix} \underline{\eta}^* \\ \underline{\eta}_o \end{bmatrix} = VL^{1/2}\,\underline{Z}^*$$

is an unconditional simulation of $\underline{\eta}$ and $\underline{\eta}_o$.

(3) A conditional simulation of $\underline{\eta}_o$, given the values of $\underline{\eta}$ is

$$\{\underline{\eta}_o, \text{ given } \underline{\eta}\} = C_{12}^{T}\, C_{11}^{-1}\, (\underline{\eta} - \underline{\eta}^*) + \underline{\eta}_o$$

KRIGING FORMULAS FOR η_o

The kriging procedure estimates each $\eta_0(t_j)$ by a linear combination of the interval of values of η_n surrounding the time t_j. The idea here is to use all values of η_n that are correlated with $\eta_0(t_j)$. A reasonable choice would be to include all η_n measured from the platform within two wave periods of the time t_j. Here a good value for the wave period would be the inverse of the frequency at the peak of the spectra. The value $\eta_0(t_j)$ is estimated by

$$\hat{\eta}_o(t_j) = \Sigma\, a_n \eta_n$$

where the summation extends over the times surrounding t_j. The coefficients, a_n, are computed to minimize

$$Q = E[\{\eta_o(t_j) - \hat{\eta}_o(t_j)\}^2]$$

subject to the constraint that

$$E[\eta_o(t_j) - \hat{\eta}_o(t_j)] = 0$$

A full discussion of this procedure is given by Borgman (1985, p. 14), a copy of which is forwarded with this report.

The computations may be summarized as follows. Let $n_1 \le n \le n_2$ be the interval of values of η_n used.

C^*_{11} = (n_2-n_1+1) by (n_2-n_1+1) covariance matrix of the η_n for $n_1 \leq n \leq n_2$

$\underline{C}_{10}$ = vector whose ℓ–th element is the covariance between $\eta_{n_1+\ell-1}$ and $\eta_o(t_j)$

$\underline{1}$ = vector of n_2-n_1+1 components, all of which equal 1.0

$\underline{a}$ = vector of coefficients to be multiplied by the η_n in the interval

The kriging equations which solve the constrained minimization are computed from

$$\begin{bmatrix} C^*_{11} & \underline{1} \\ \underline{1}^T & 0 \end{bmatrix} \begin{bmatrix} \underline{a} \\ \lambda \end{bmatrix} = \begin{bmatrix} \underline{C}_{10} \\ 1.0 \end{bmatrix}$$

where, λ is the Lagrangean multiplier imposing the constraint. Thus

$$\begin{bmatrix} \underline{a} \\ \lambda \end{bmatrix} = \begin{bmatrix} C^*_{11} & \underline{1} \\ \underline{1}^T & 0 \end{bmatrix}^{-1} \begin{bmatrix} \underline{C}_{10} \\ 1.0 \end{bmatrix}$$

The mean–square–error of the estimate of $\eta_o(t_j)$ is

$$\text{mean square error} = E[\{\eta_o(t_j) - \hat{\eta}_o(t_j)\}^2] = \sigma^2 - \lambda - \underline{a}^T \underline{C}_{10}$$

where

$$\sigma^2 = 2\int_0^\infty S(f)df$$

from the sea surface spectral density.

SUMMARY AND CONCLUSIONS

1. Kriging and conditional simulation algorithms have been developed for estimating the water level elevation at a fixed reference location from the measured water level locations on the moving platform.

2. Both kriging and conditional simulation have straightforward mathematical formulas, once the appropriate covariance matrices have been computed.

3. An algorithm based on a Bessel function series and the use of the fast Fourier Transform to compute a subsidiary function, $R_n(\rho, \tau)$, is derived. This is the main work in the report. Both kriging, and conditional simulation procedures are completely derived in cited references.

4. Recommendations are given for procedures to compute the Bessel functions in the formula and the other aspects of the algorithm.

5. The kriging procedure will probably be the best choice of the geostatistical techniques for use in estimating the water level elevations at the reference location. A convenient error measure arises naturally from the computations and the procedure is relatively rapid to compute.

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APPENDIX

The equivalences between the spreading function models are most easily derived from the half–peak width of the functions. The half–peak with of D1θ) is two times the $(\theta - \mu)$ value at which

$$D(\theta_*) = 0.5\, D(\mu)$$

Let Δ_{HP} be this value

$$\Delta_{HP} = 2(\theta_* - \mu)$$

For the generalized cosine–squared model

$$D(\mu) = c$$

so the equation becomes

$$c \cos^{2s}[(\theta_* - \mu)/2] = 0.5c$$

$$\cos[(\theta_* - \mu)/2] = (0.5)^{1/(2s)}$$

$$\Delta_{HP}^{CS} = 2(\theta_* - \mu) = 4 \text{ arc } \cos[0.5^{1/(2s)}]$$

For the von Mises model

$$D(\mu) = 1/\{2\pi I_0(a)\}$$

Hence the equation is

$$e^{a\cos(\theta_*-\mu)} = 0.5$$

$$a\cos(\theta_*-\mu) = \text{Log}(0.5)$$

$$\Delta_{\text{HP}}^{\text{VM}} = 2(\theta_*-\mu) = 2 \text{ arc cos}\left\{ \frac{\text{Log}(0.5)}{a} \right\}$$

Finally, for the wrapped–normal model, in the case most common in ocean wave work where

$$D(\theta) \approx \frac{e^{-(\theta-\mu)^2/(2\sigma^2)}}{\sqrt{2\pi}\,\sigma}$$

$$D(\mu) = \frac{1}{\sqrt{2\pi}\,\sigma}$$

and the equation to be solved is

$$e^{-(\theta_*-\mu)^2/(2\sigma^2)} = 0.5$$

$$\frac{-(\theta_*-\mu)^2}{2\sigma^2} = \text{Log}(0.5)$$

$$(\theta_*-\mu)^2 = -2\sigma^2\text{Log}(0.5)$$

$$\Delta_{\text{HP}}^{\text{WN}} = 2(\theta_*-\mu) = 2\sqrt{-2\sigma^2\text{Log}(0.5)}$$

Thus, a reasonable equivalence between parameters is given by

$$2 \text{ arc cos}\left[0.5^{1/(25)}\right] = \text{arc cos}\left[\frac{\text{Log}(0.5)}{\text{a}}\right] = \sigma\sqrt{-2\text{Log}(0,5)}$$

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