ESTIMATION OF IRREGULARITY FACTOR FROM A POWER SPECTRUM*

Pei-Ning Li**
Yi-Wen Cheng
Fracture and Deformation Division
National Bureau of Standards
Boulder, Colorado 80303

Abstract

Several simplified techniques for estimating the irregularity factor of a power spectrum are presented.

Key Words: bandwidth, irregularity factor, power spectrum, random fatigue

**Guest worker, on leave from East-China Institute of Chemical Technology, Shanghai, China

1. INTRODUCTION

In many structural applications fatigue under random loading is a major problem, such as offshore platforms in the North Sea environments. The record of the load-time history under random loading is usually so long that it is difficult to read or described it or to recognize patterns. If the random loading is a stationary and Gaussian process, then there exists a power spectrum, G(f), which possesses all the statistical properties of the original load-time history [1]. The power spectrum is, therefore, usually used as to represent the random load-time history.

Several important parameters in the random loading fatigue analysis can be derived from the power spectrum. These parameters include the root-mean-square value of the load amplitude, the average rise and fall, and the irregularity factor, α [2-3]. In this paper, simplified techniques for estimating α from a given power spectrum are presented.

2. IRREGULARITY FACTOR

The irregularity factor, α, is defined as the ratio of the number of zero crossings to the number of peaks in a load-time history.

$$\alpha = N_o/F_o \tag{1}$$

where N_o is the number of zero crossings and F_o is the number of peaks. The exact values of N_o and F_o can be evaluated from G(f) as follows:

$$N_o = (M_2/M_o)^{1/2} \tag{2a}$$

$$F_o = (M_4/M_2)^{1/2} \tag{2b}$$

where M_o, M_2, and M_4 are the zeroth, second and fourth moments of G(f) about the origin (zero frequency), and defined as:

$$M_0 = \int_0^\infty G(f)\, df \tag{3a}$$

$$M_2 = \int_0^\infty f^2\, G(f)\, df \tag{3b}$$

$$M_4 = \int_0^\infty f^4\, G(f)\, df$$

where f is frequency, thus,

$$\alpha = M_2/(M_0 M_4)^{1/2} \tag{4}$$

The irregularity factor, α, not only describes the irregularity of the random load-time history but also is a measure of the bandwidth of G(f). As α approaches unity, the distribution of the loading peaks approximates to the Rayleigh distribution [1], and the shape of G(f) is sharply peaked at the center frequency or far away from the origin and it is called narrow band. A single-frequency sine wave loading can be described as a Dirac-delta function power spectrum. The value of α decreases with increasing the width of the power spectrum. As α approaches zero the power spectrum is uniform all over the frequencies; this is the wide-band limit.

3. CALCULATION OF IRREGULARITY FACTOR FROM A POWER SPECTRUM

3.1 Direct Integration of Power Spectral Density Function

The value of α can be evaluated from equation (4) by integrating equations (3a), (3b), and (3c). The integrations can become tedious and time consuming if the shape of G(f) is irregular. One simplified way of evaluating α is to break G(f) into several simpler geometries such as those shown in Figure 1, and then to evaluate the moments according to the following equations:

$$M_j = \frac{1}{(j+1)(j+2)} \sum_{i=1}^{n} (G_i - G_{i+1}) \frac{f_{i+1}^{(j+2)} - f_i^{(j+2)}}{f_{i+1} - f_i}, \qquad j = 0, 2, 4 \tag{5}$$

where G_i's and f_i's are power spectral densities and frequencies respectively, as shown in Figure 1. The derivation of equation (5) is given in the Appendix.

For example, using equation (5) and the break-up diagram shown in Figure 2, the calculated value of α is 0.699. The more precise value α is 0.697. The error of using equation (5) is only 0.14 percent.

3.2 Estimated from Characteristic Width and Center Frequency of a Power Spectrum

For the case of rectangle power spectra a simple expression for α can be derived mathematically from Equation (4), and stated in terms of the width, W, and the center frequency, f_c, as follows [4]:

$$\alpha = \sqrt{\frac{5(9+6B^2+B^4)}{9(5+10B^2+B^4)}} \tag{6}$$

where $B = W/2f_c$ (7)

Here, B is a directly perceived dimensionless parameter of the power spectrum, and is defined as the geometric dimensionless bandwidth. The physical meanings of W and f_c are obvious in this case, and both can be determined in a straightforward manner. For irregular spectra, this approach may also be a promising method for calculating α, if the appropriate values of W and f_c can be determined such that Equation (6) still provides good estimates of α.

3.2.1 Determination of Characteristic Width, W

The center frequency of any single-peaked, symmetric power spectrum is, obviously, at the middle of the frequency range. Equation (8) is suggested for the calculation of the characteristic width, W.

$$W = \bar{W}\sqrt{\frac{A_{rec}}{A_{p.s.}}} \tag{8}$$

where A_{rec} is the area of a rectangle enveloping the power spectrum, $A_{p.s.}$ is the area of the power spectrum, and $\bar{W}$ is the arithmetic average width of the power spectrum. Then, for example, the value of $\bar{W}$ for an isosceles triangular power spectrum is half of its base width, $A_{rec}/A_{p.s.}$ is equal to 2, and its irregularity factor, α, can be derived from Equation (4). Written in the form of Equation (9), it is

$$\alpha = \sqrt{\frac{5(9+6B^2+B^4)}{9(5+10B^2+\frac{4}{3}B^4)}} \tag{9}$$

which is almost the same as Equation (6).

The results of numerous calculations for several shaped power spectra, including isosceles triangles, rectangles, isosceles trapezoids and pagodas, are plotted in Figure 3 in the form of α vs. B. All values of α in all figures in this paper are calculated by the theoretical Equation (4). It can be seen that the α vs. B curves of four kinds of power spectra are coincident, demonstrating that the estimate of W by Equation (8), combined with Equations (6) and (7), does in fact provide good estimates of α.

3.2.2 Determination of Center Frequency, f_c

The α of an arbitrary triangular power spectrum with a fixed base (i.e. with a constant characteristic width) varies with the location of the peak within the width, because of the changing of the center frequency. Several candidates for center frequency, including the frequency at the peak of the power spectrum, the frequency at the center of gravity of the power spectrum, the frequency at the middle of the frequency range, and the frequency at the middle of the half height width, have been checked. It is recommended that the frequency at the middle of the half-height width of the power spectrum be taken as the center frequency, because this choice gives smallest scatter among the α vs. B curves of several typical asymmetric single-peaked power spectra. The α vs. B curves of right triangular power spectra with the right angle to the left or right are compared with the α vs. B curve of rectangular power spectra in Figure 4. It is demonstrated that this estimate of f_c, combined with Equations (6) and (7), provides a good estimate of α.

3.2.3 Behavior of Double-peaked Power Spectra

The irregularity factor, α, of a double-peaked power spectrum consisting of two very narrow rectangles is derived from Equation (4)

$$\alpha = \frac{1 + A F^2}{\sqrt{(1+A)(1+AF^4)}} \tag{10}$$

$$A = A_2 / A_1 \tag{11}$$

$$F = f_{c2} / f_{c1} \tag{12}$$

where A_1, A_2, f_{c1} and f_{c2} are the areas and center frequencies of each rectangles, respectively.

For double-peaked power spectra consisting of two wide peaks, Equation (10) is still a good estimate. The estimation error of Equation (10) is within ±5 percent for power spectra consisting of two wide rectangles or two isosceles triangles in a large range of A = 0.1 to 10, F = 2.333 to 7, G_2/G_1 = 0.1 to 10, and α = 0.96 to 0.56, shown in Table 1.

Equation (10) has been plotted in Figure 5. It is shown that the larger the F and the smaller the A, the smaller the irregularity factor, α. As $A_1 << A_2$, the irregularity factor, α, is dominated by the second peak. However, as $A_1 >> A_2$, the second peak may still play an important role in determination of α, because the second peak with higher frequency plays a more significant role in M_4 and M_2.

Two peaks of a practical double-peaked power spectrum are usually connected to each other at their bases. Therefore the double-peaked power spectrum is cut into two parts and their center frequencies and areas are estimated. Then the irregularity factor, α, can be calculated by Equation (10). For example, the North Sea power spectrum shown in Figure 6 is cut at 0.3 Hz, 0.25 Hz, or 0.2 Hz, and modified to two triangles, ΔABC and ΔDEF, or $\Delta ABC'$ and $\Delta D'EF$, or $\Delta ABC''$ and $\Delta D''EF$. All triangles have the same areas of the original peaks. The F and the A obtained are 2.70 and 0.1514, 2.71 and 0.2278, or 2.74 and 0.3165. The estimated α by Equation (10) is 0.6520, 0.6616, or 0.6785. The error comparing with the accurate calculation is -6.4, -5.07, or -2.65 percent. Since the second peak plays more important roles in M_4 and M_2, making the second peak area larger gets more accurate estimation of α.

4. CONCLUSION

Several simplified methods on the estimation of the irregularity factors of power spectra are presented.

For single-peaked power spectra, the irregularity factors can be rapidly estimated by Equation (6), a simple function of geometric dimensionless bandwidth, B, which is a directly perceived measure of the power spectrum

The behavior of the double-peaked power spectra has been discussed. Their irregularity factors can be rapidly estimated by Equation (10).

Equation (5) is recommended for more accurate calculation of the irregularity factor for any kind of power spectra in a short time.

REFERENCES

1. Bendat, J. S., "Principles and Application of Random Noise Theory", John Wiley & Sons, Inc., 1958.

2. Smith, S. H., "Fatigue Crack Growth Under Axial Narrow and Broad Band Random Loading," in <u>Acoustical Fatigue in Aerospace Structure</u>, 1964, pg. 331-360.

3. Beer, F., Wagner, R., Bahar, L., and Ravera, R., "On the Statistical Distribution of Rise and Falls in a Stochastic Process," Lehigh Institute of Research Progress Report, 1961. (Fracture Mechanics Research for the Boeing Airplane Company).

4. Pook, L. P., "Proposed Standard Load Histories for Fatigue Testing Relevant to Offshore Structures," National Engineering Laboratory, NEL Report No. 624, Oct. 1976.

APPENDIX

A broken line can be used instead of the original curve of any kind of power spectra. Figure 1 shows a single-peaked power spectrum (dash line), as an example, represented by a broken line consisting of four segments (solid lines).

$$G(f) = (G_2 - 0)\frac{f - f_1}{f_2 - f_1}$$ for first segment

$$G(f) = (G_3 - G_2)\frac{f - f_2}{f_3 - f_2} + G_2$$ for second segment

$$G(f) = (G_4 - G_3)\frac{f - f_4}{f_4 - f_3} + G_4$$ for third segment

$$G(f) = (0 - G_4)\frac{f - f_5}{f_5 - f_4}$$ for fourth segment

therefore

$$M_0 = \frac{1}{2}\Big[(0 - G_2)\frac{f_2^2 - f_1^2}{f_2 - f_1} + (G_2 - G_3)\frac{f_3^2 - f_2^2}{f_3 - f_2} + (G_3 - G_4)\frac{f_4^2 - f_3^2}{f_4 - f_3} + (G_4 - 0)\frac{f_5^2 - f_4^2}{f_5 - f_4}\Big] \qquad (A-1)$$

$$M_2 = \frac{1}{12}\Big[(0 - G_2)\frac{f_2^4 - f_1^4}{f_2 - f_1} + (G_2 - G_3)\frac{f_3^4 - f_2^4}{f_3 - f_2} + (G_3 - G_4)\frac{f_4^4 - f_3^4}{f_4 - f_3} + (G_4 - 0)\frac{f_5^4 - f_4^4}{f_5 - f_4}\Big] \qquad (A-2)$$

$$M_4 = \frac{1}{30}\left[(0-G_2)\frac{f_2^6-f_1^6}{f_2-f_1} + (G_2-G_3)\frac{f_3^6-f_2^6}{f_3-f_2} + (G_3-G_4)\frac{f_4^6-f_3^6}{f_4-f_3} + (G_4-0)\frac{f_5^6-f_4^6}{f_5-f_4}\right] \qquad (A-3)$$

Rewrite (A - 1), (A - 2) and (A - 3) in a general format:

$$M_j = \frac{1}{(j+1)(j+2)} \sum_{i=1}^{n} (G_i - G_{i+1}) \frac{f_{i+1}^{(j+2)} - f_i^{(j+2)}}{f_{i+1} - f_i}, \qquad (j=0,2,4) \qquad (5)$$

which can be used for double-peaked power spectra as well as single-peaked one.

Table 1 Estimated errors of irregularity factor of double-peaked power spectra

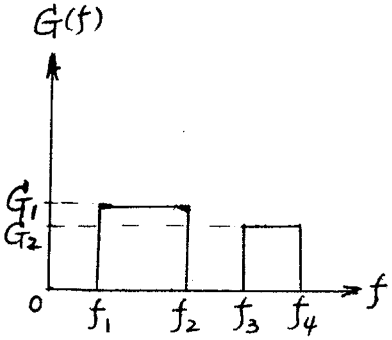
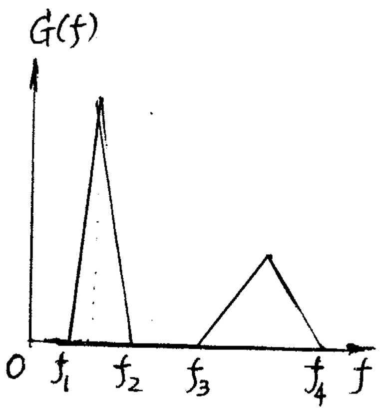
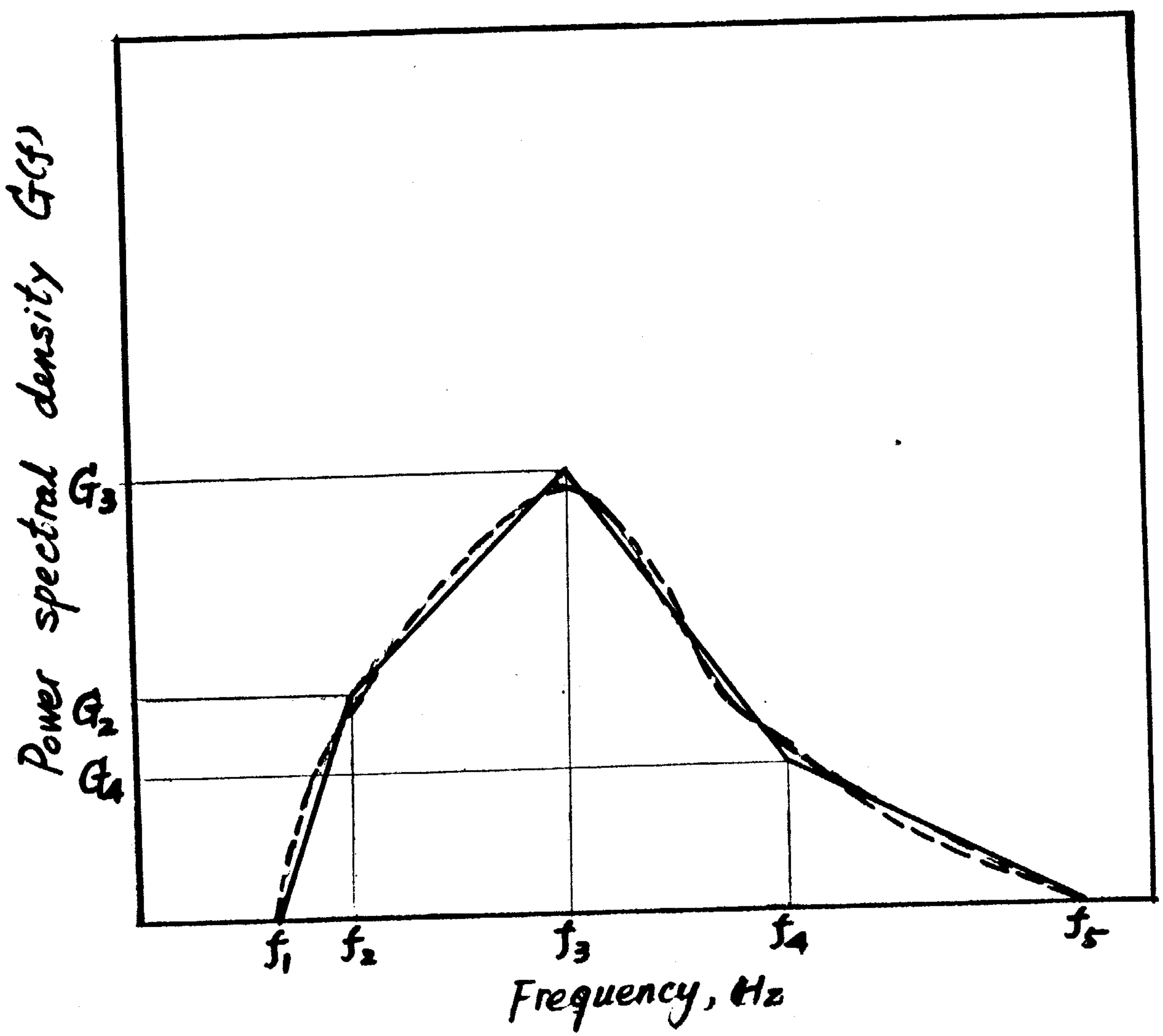
Shape of Spectra	$G_1:G_2$	$f_1:f_2:f_3:f_4$	F	A	Value of α by Eq. (5)	Value of α by Eq. (17)	error %
	1 : 1	0 : 1 : 2 : 3	5	1	0.7240	0.7348	1.50%
		1 : 2 : 3 : 4	2.333	1	0.8138	0.8232	1.16%
		0 : 1 : 3 : 4	7	1	0.7162	0.7213	-0.71%
		1 : 2 : 4 : 5	3	1	0.7762	0.7808	0.60%
	1 : 2	0 : 1 : 2 : 3	5	2	0.8160	0.8325	2.02%
		1 : 2 : 3 : 4	2.333	2	0.8733	0.8840	1.23%
		0 : 1 : 3 : 4	7	2	0.8164	0.8247	1.02%
		1 : 2 : 4 : 5	3	2	0.8532	0.8592	0.70%
	2 : 1	0 : 1 : 2 : 3	5	0.5	0.6192	0.6225	0.53%
		1 : 2 : 3 : 4	2.333	0.5	0.7559	0.7641	1.08%
		0 : 1 : 3 : 4	7	0.5	0.6000	0.6007	0.12%
		1 : 2 : 4 : 5	3	0.5	0.6939	0.6971	0.46%
	5 : 1	0 : 1 : 2 : 3	5	0.2	0.4969	0.4880	-1.79%
		1 : 2 : 3 : 4	2.333	0.2	0.7161	0.7244	1.16%
		0 : 1 : 3 : 4	7	0.2	0.4559	0.4494	-1.43%
		1 : 2 : 4 : 5	3	0.2	0.6150	0.6163	0.21%
	1 : 5	0 : 1 : 2 : 3	5	5	0.8989	0.9200	2.35%
		1 : 2 : 3 : 4	2.333	5	0.9312	0.9435	1.32%
		0 : 1 : 3 : 4	7	5	0.9057	0.9165	1.19%
		1 : 2 : 4 : 5	3	5	0.9249	0.9320	0.77%
	10 : 1	0 : 1 : 2 : 3	5	0.1	0.4384	0.4188	-4.47%
		1 : 2 : 3 : 4	2.333	0.1	0.7280	0.7396	1.59%
		0 : 1 : 3 : 4	7	0.1	0.3756	0.3623	-3.54%
		1 : 2 : 4 : 5	3	0.1	0.5996	0.6298	5.0%
	1 : 10	0 : 1 : 2 : 3	5	10	0.9342	0.9572	2.46%
		1 : 2 : 3 : 4	2.333	10	0.9568	0.9693	1.31%
		0 : 1 : 3 : 4	7	10	0.9434	0.9554	1.27%
		1 : 2 : 4 : 5	3	10	0.9559	0.9635	0.80%

Table 1 continued



5 : 1	1 : 3 : 4 : 8	3	0.4	0.6400	0.6727	5.11%
	1 : 3 : 5.5 : 6.6	3	0.1	0.6314	0.6005	-4.89%
1 : 3	0 : 1 : 1 : 2	3	3	0.8687	0.8963	3.18%
	1 : 3 : 3 : 5	2	3	0.9106	0.9285	1.97%
	1 : 2 : 2 : 3	1.666	3	0.9370	0.9495	1.33%
1 : 2	0 : 1 : 1 : 2	3	2	0.8341	0.8592	3.01%
	1 : 3 : 3 : 5	2	2	0.8872	0.9045	1.95%
	1 : 2 : 2 : 3	1.666	2	0.9214	0.9335	1.31%
1 : 1	0 : 1 : 1 : 2	3	1	0.7612	0.7810	2.60%
	1 : 3 : 3 : 5	2	1	0.8412	0.8575	1.94%
	1 : 2 : 2 : 3	1.666	1	0.8924	0.9050	1.41%
	0 : 1 : 2 : 3	5	1	0.7167	0.7348	2.53%
	1 : 2 : 3 : 4	2.333	1	0.8186	0.8232	1.31%
2 : 1	0 : 1 : 1 : 2	3	½	0.684	0.697	1.90%
	1 : 3 : 3 : 5	2	½	0.8006	0.8165	2.00%
	1 : 2 : 2 : 3	1.666	½	0.872	0.885	1.49%
	0 : 1 : 2 : 3	5	½	0.6208	0.6225	0.27%
	1 : 2 : 3 : 4	2.333	½	0.7600	0.7640	0.52%
3 : 1	0 : 1 : 1 : 2	3	1/3	0.6455	0.6547	1.43%
	1 : 3 : 3 : 5	2	1/3	0.7863	0.8030	2.10%
	1 : 2 : 2 : 3	1.666	1/3	0.8680	0.8828	1.71%
	0 : 1 : 2 : 3	5	1/3	0.5593	0.5587	-0.11%
	1 : 2 : 3 : 4	2.333	1/3	0.7275	0.7389	1.57%

fig. 1



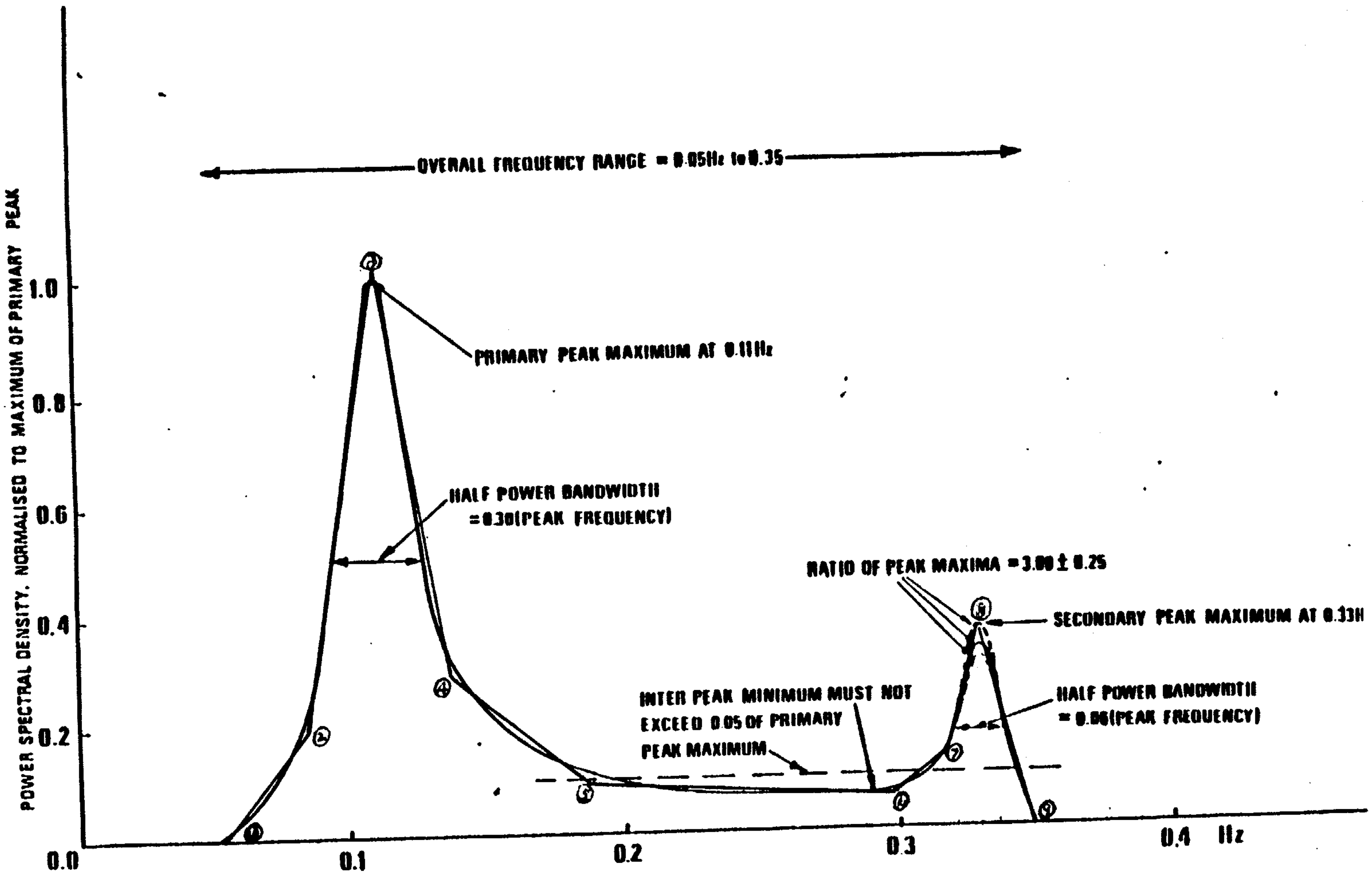


Fig. 2 SPECIFICATION OF POWER SPECTRUM

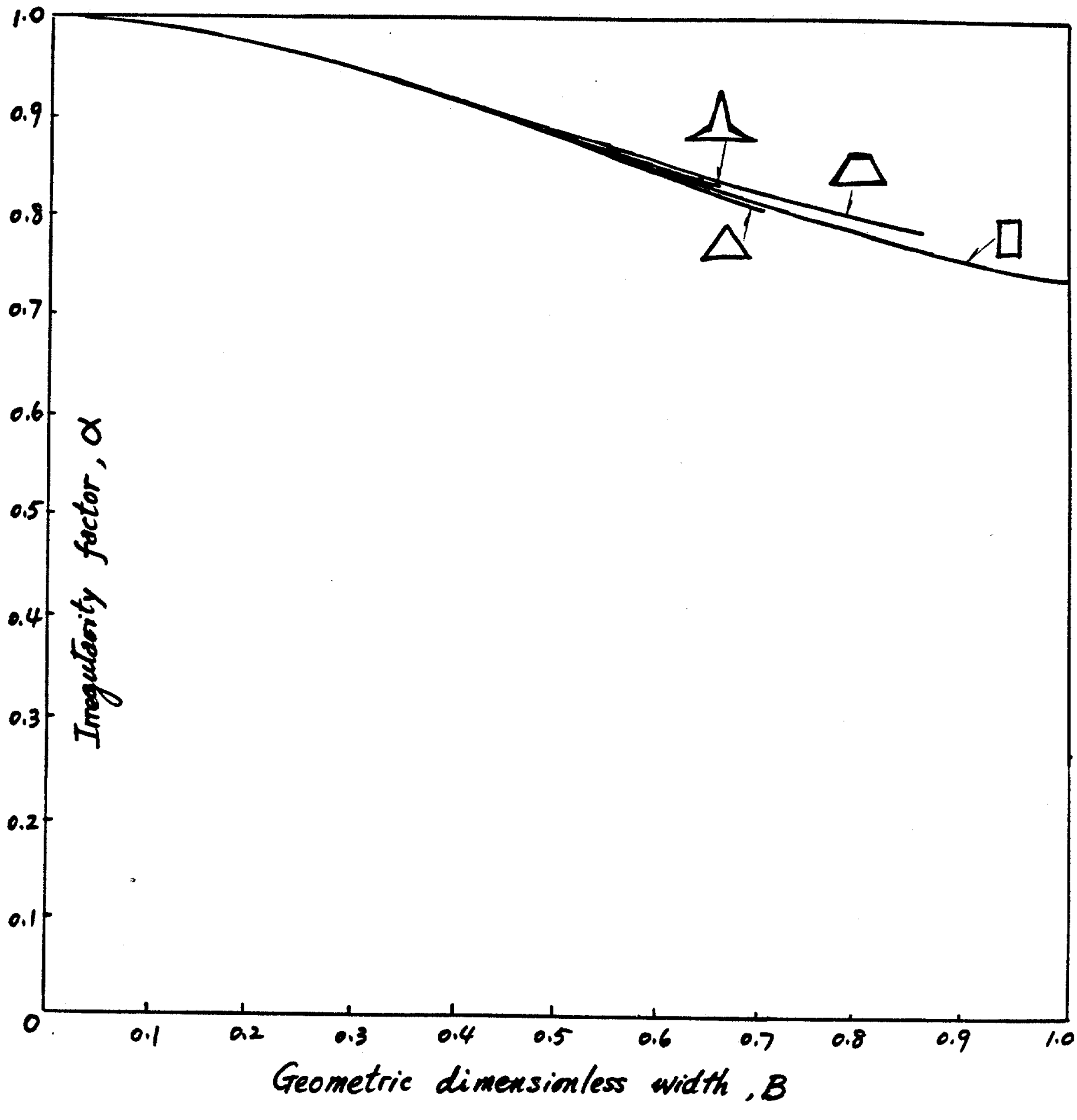
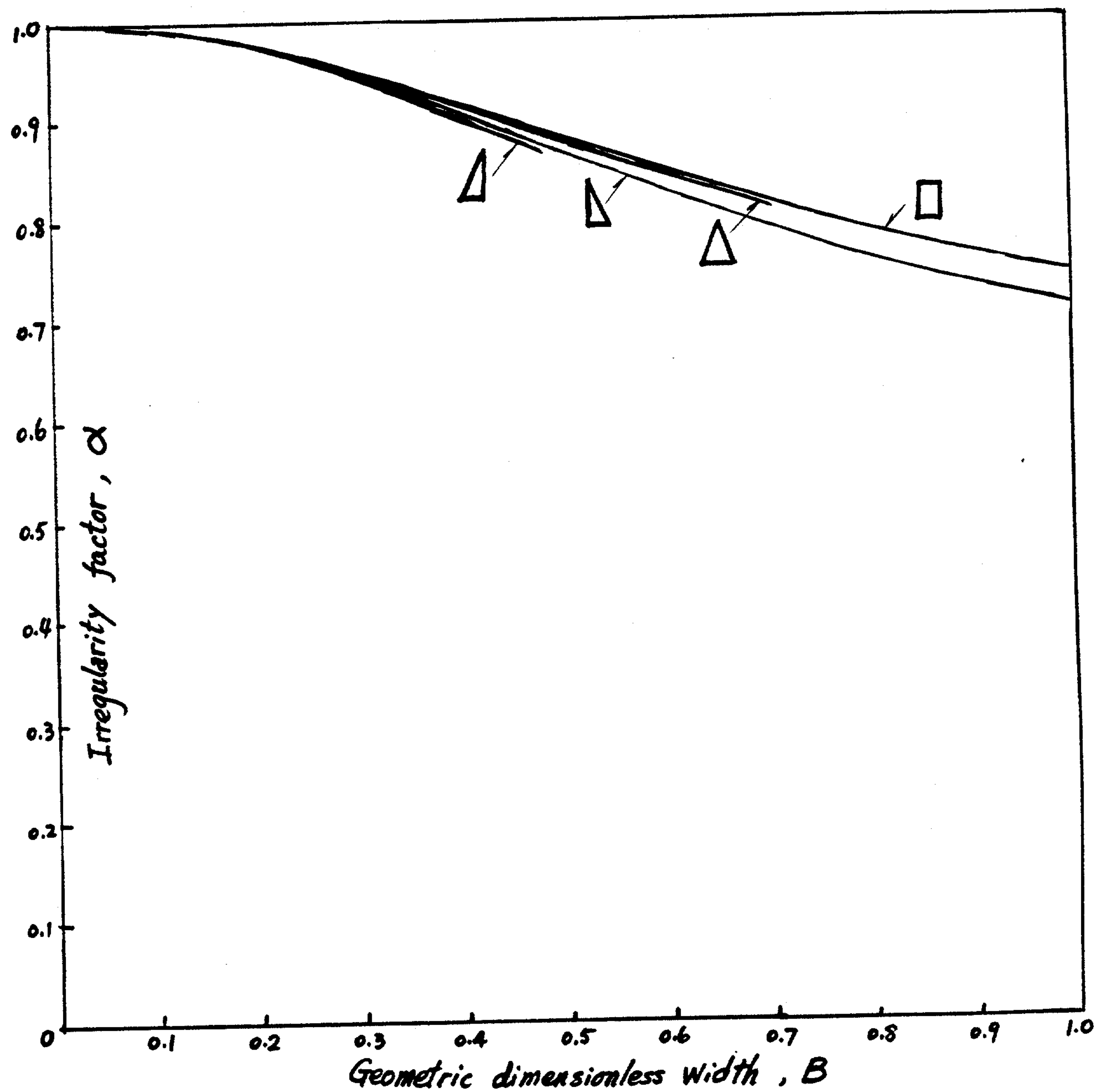
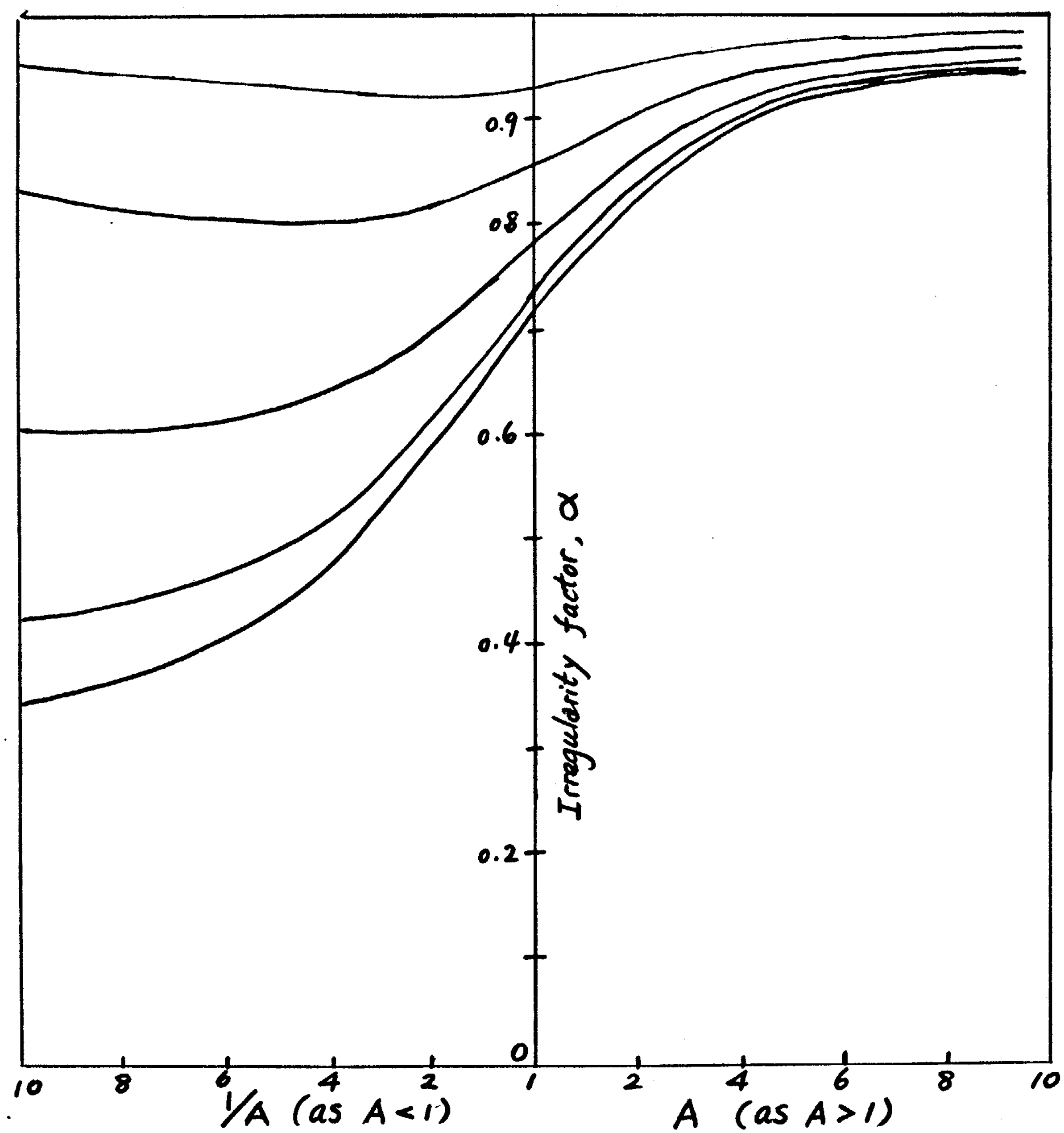

1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
Irregularity factor, α
Geometric dimensimless width, B

Fig. 3

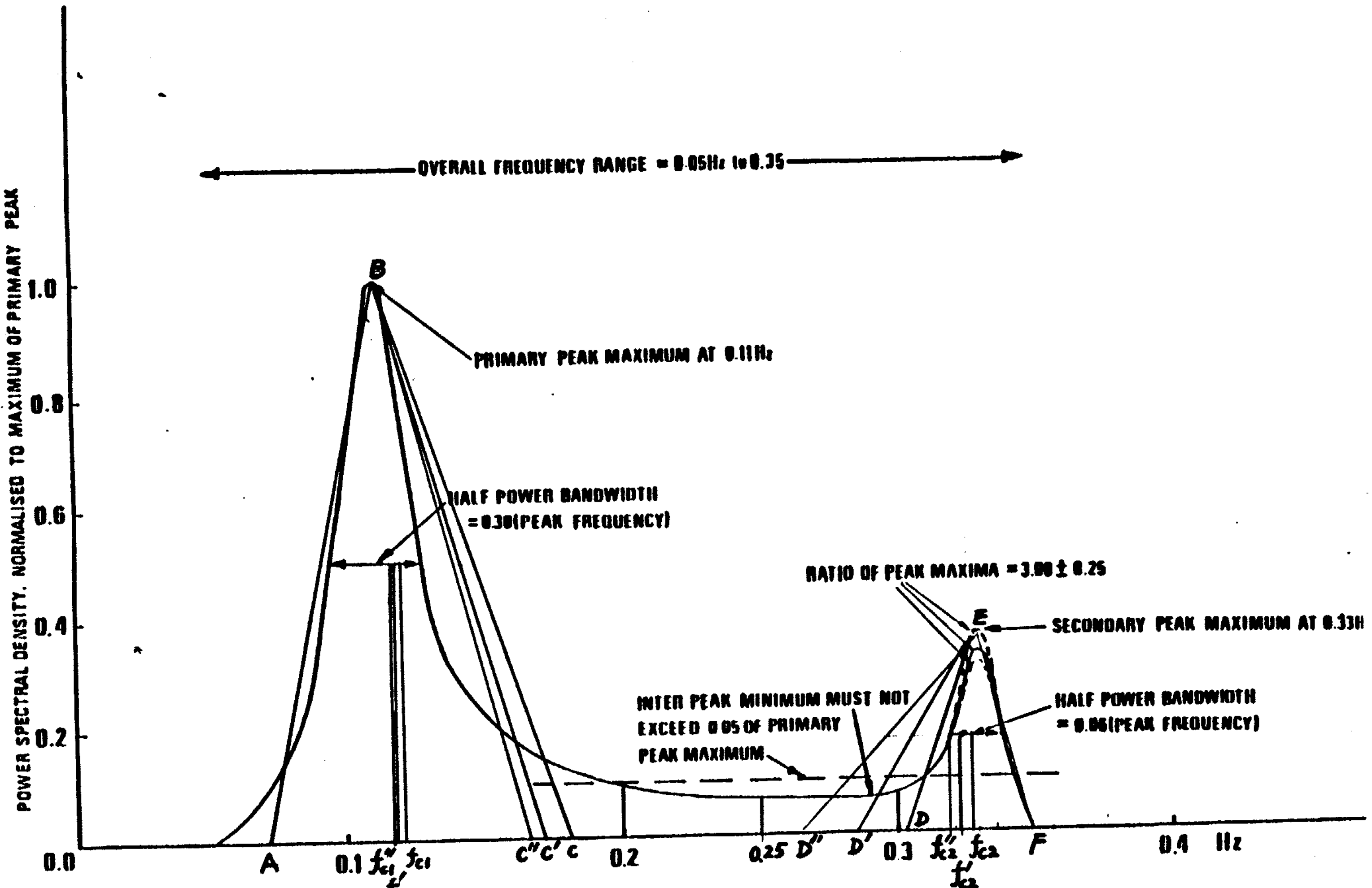
fig. 4


1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
Irregularity factor, α
Geometric dimensionless width, B



Irregularity factor, α
0.9
0.8
0.6
0.4
0.2
0
10
8
6
4
2
1
2
4
6
8
10
1/A (as A < 1)
A (as A > 1)

fig. 5



SPECIFICATION OF POWER SPECTRUM

figure. 6