Attachment I-1

MECHANICAL PROPERTIES OF SEA ICE

Theoretical phase, September 1980 - November 1981

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Contents

Preamble

This is an integrated account of the theoretical investigations described in the technical reports M1 - M8 prepared during the year. While a little of the less interesting calculation has been omitted, the main purpose is to provide a comprehensive, self-contained, account of the theoretical development which must complement the experimental programme. On the basis of existing qualitative knowledge of the response of ice to load, we must ask what classes of constitutive model (constitutive law) are compatible with this behaviour, and analyse the response of such models in feasible test geometries to correlate model functions with test data. An acceptable model must describe the response to general (multi-axial) stress in a three-dimensional world, and must satisfy the basic invariance principles of physics. Subject to these requirements we want to determine, at least as a preliminary objective, the minimal complexity necessary in each model class to predict the known shape of response, and then to formulate a minimal, feasible, test programme necessary to determine the model functions.

While models so constructed exhibit identical response to the test loads, distinct responses to other load configurations can be expected, and further test data and field observations will be necessary to distinguish the merits of different models, and possibly indicate additional features which should be included. Furthermore, different models may be better approximations in different applications, and it is important to identify the dominant response features in prescribed applications to make the best choice of model. Distinct simpler models may offer considerable advantage over a more general model with wider validity when numerical solutions of complex engineering problems are required. The essential feature of ice response on engineering time scales (excluding very short time dynamic effects) is its non-linear transient creep, which requires a non-linear viscoelastic model. Even though strains may remain small in many applications, the superposition property of linear viscoelastic models does not hold to any acceptable approximation, and this very attractive tractable linear theory must be rejected. A long-time scale non-linear viscous model conventional in glaciology ignores the crucial transients. The first phase of the Project has focussed on the recently developed non-linear viscoelastic relations of differential type, distinct forms for fluids and for solids. The latter allow strain jumps when stress jumps are applied, allow anisotropic configurations, and exhibit induced anisotropy in configurations reached by distortion of an initially isotropic state. Many positive conclusions are reached in regard to the questions raised above. There is a final brief commentary on the merits and difficulties associated with non-linear integral operator relations, which should be investigated in the next phase.

1. Introduction

Present knowledge of the transient response of ice to applied load under controlled conditions is limited to results of laboratory tests in uni-axial compressive stress, which must therefore be the starting point of the construction of constitutive models. While the quality, or shape, of the responses to constant stress and to constant strain-rate loading has been established, many more detailed quantitative results must be accurately determined to complete the uni-axial stress description. Such experiments are a preliminary stage of the present Project, and will also serve to test and assess instrumentation and technique before embarking on the essential two-dimensional test programmes discussed later. A heuristic approach is to deduce minimal sets of physical variables necessary in a constitutive relation (of a given class) to exhibit the known shapes of these responses in uni-axial stress, which was the basis of the recent development of viscoelastic fluid and solid relation of differential type (Morland 1979, Morland and Spring 1981, Spring and Morland 1981). Since this account draws substantially from the theory described in these papers, it is convenient to introduce the abbreviated reference notations (M), (MS), (SM), respectively.

X The rational procedure, followed in these papers, is to construct relations between frame-ind~~e~~ifferent tensors which are three-dimensional measures of the physical variables required in the minimal sets. In the above models, stress, strain-rate, and

stress-rate and strain or strain-acceleration are used. Thus the fundamental physics invariance principles - coordinate invariance and material frame indifference (objectivity) - are immediately satisfied. Such tensor measures are not unique, and a general algebraic relation between the tensors is far too arbitrary to correlate with constant stress and constant strain-rate responses. So simplified forms and restricted dependence of the response coefficients, scalar coefficients of the different tensors, are proposed, retaining only the flexibility needed to describe the known response shapes. There are clearly a variety of alternative relations of a given class which could be correlated with the same uni-axial data, but which will predict distinct responses under different loading geometries and different load histories. A successful programme of multi-axial load testing is required before such distinctions can be constructively evaluated. Restricting the above differential tensor relations to uni-axial stress gives single relations between the uni-axial stress and stress-rate, and axial strain-rate and strain-acceleration or strain. However, tensor terms combine into single uni-axial terms, and so cannot be distinguished by uni-axial response, and response coefficients which are functions of various tensor invariants appear only as functions of the axial variables. In consequence a model can only be completed by determining the responses in suitable multi-axial load geometries. A particular two-dimensional test geometry has been analysed under the

the Project (M2), and shown to yield the required number of independent relations. There is also a consideration of domains of dependence in the space of stress tensor invariants, which can be covered by tests involving compressive stresses alone.

While the restriction of valid tensor relation to uni-axial stress is the natural order, to highlight the physical description and heuristic process of model construction we will concentrate first on the uni-axial response and the uni-axial relations which are derived from valid tensor relations (MS, SM). The correlation of constant stress and constant strain-rate responses with coefficients in the uni-axial relations is described, and the distinction between fluid and solid model correlation noted. In particular it is seen that the two separate test responses cannot be independent if described by this fluid model, but are not sufficient to determine the uni-axial coefficients of the solid model. A general analysis of the solid model response under the Project (M6, M7) has shown that no uni-axial load test can yield a third independent relation necessary to determine the complete set of uni-axial coefficients. Reduced forms which allow determination have been investigated (M8), together with their implication for, and recognition by, different response features. A particular form has been chosen for correlation with uni-axial data obtained in the preliminary phase of the experimental programme.

Next there is a brief description of the fluid and solid model tensor relations, to show the increased number of response coefficients which arise, and their dependence on various tensor invariants. The response to general tri-axial (principal) stress is analysed, and two-independent stress geometries are investigated (M2) to learn how two independent relations in addition to the incompressibility constraint can be obtained to complete the three-dimensional description. Explicit terminology is introduced to define and distinguish different two-stress geometries, to eliminate previous ambiguous (at least to the writer) descriptions. A bi-axial stress geometry is shown to yield the required independent relations. In addition, the domains of stress tensor invariants space covered by compressive stresses alone in different loading geometries are determined. Dependence on shear stress invariants or on one shear invariant and pressure (M5) present different pictures. The implications of relaxing the conventional incompressibility approximation and introducing compressibility or dilatancy relations are discussed (M3).

A preliminary report (M1) outlined various objectives and plans for the Project. The major element is the experimental programme. In this first period we have focussed on a theory for constant temperature conditions and (initially) isotropic material, and disregarded effects of salinity. While anisotropy and salinity effects are known to be significant, existing data is inadequate to formulate a satisfactory theory. It is possible that plausible idealised models may have to be devised in advance of experiments, in order to assess likely effects and design optimal experiments.

The marked dependence of creep-rate on temperature is well established. Test data must be obtained at a sufficient number of constant temperatures in the range 230K - 273K of practical interest to infer and correlate the role of temperature. At present, the most attractive theory is that of the "thermorheologically simple material" in which temperature influences only rates of change, which implies that a universal (temperature-independent) mechanical relation holds on a pseudo or reduced time scale (Morland and Lee 1960). Applied originally to linear viscoelastic materials, it is readily extended to non-linear viscoelastic differential relations (M), and involves only the introduction of a single scalar "time-shift function" of temperature. In the conventional viscous fluid relations adopted in glaciology, this shift function is the usual temperature dependent coefficient. It has yet to be confirmed that this single rate factor applies over the complete transient response, coincident with the minimum strain-rate factor of the viscous relation.

2. Uni-axial stress response

The basis of our constitutive model construction is the qualitatively established constant stress and constant strain-rate responses in uni-axial compressive stress. A wide account of the mechanical properties of ice, including these features, is given in a recent review article (Mellor 1980). If ℓ_o is the initial specimen length at time $t = 0$, and ℓ its length at time t, then the longitudinal engineering stress e measured as contraction per unit initial length (positive in compression) is

$$e = \frac{\ell_o - \ell}{\ell_o}\ , \quad < 1 \, . \tag{2.1}$$

The engineering strain-rate $\dot{e}$, and natural strain- rate r measured as rate of decrease of length per unit current length, are given and related by

$$\dot{e} = -\dot{\ell}/\ell_o\ , \quad \dot{r} = -\dot{\ell}/\ell\ , \qquad \dot{e} = (1-e)r\ . \tag{2.2}$$

The rate $\dot{e}$ with its reference to the initial length ℓ_o is natural in a solid description, and the rate r with respect to current length is natural to a fluid description. They are approximately the same for the small strains arising in experimental data and many applications, but differ significantly in the long-time creep behaviour assumed to complete the viscoelastic continuum description. The present models contain no criteria for rupture. Also Mellor remarks that reported constant strain-rate experiments may mean constant r or constant $\dot{e}$; the latter is better described as constant (end) displacement rate.

Let σ denote the compressive axial Cauchy stress, inward force per unit current cross-section, and $\overline{\sigma}$ the nominal stress, inward force per unit initial cross-section. Constant stress could refer to either constant σ or constant $\overline{\sigma}$, but here the precise description, constant load, will be used to denote constant $\overline{\sigma}$. The present theory is developed with the incompressibility approximation, and then

$$\sigma = \overline{\sigma}(1-e)\ . \tag{2.3}$$

Interpreting the responses described by Mellor as constant load (constant $\bar{\sigma}$) and constant displacement-rate (constant $\dot{e}$) results gives the typical curves shown in Figs 1 and 2 respectively. At constant $\bar{\sigma}$ there is an initial elastic strain jump $e_e(\bar{\sigma})$ given by

$$e_e(\bar{\sigma}) = \bar{\sigma}/E_o , \tag{2.4}$$

where E_o is the Young's modulus at zero stress. It is supposed that the stress jump $\bar{\sigma}$ from zero stress is applied instantaneously at $t = 0$, and that the creep curve as $t \to 0+$ is a smooth backward extrapolation of the response after wave effects have decayed. Now E_o is of order 10^{10}Nm^{-2} (Sinha 1978, Michel 1978, Mellor 1980), so that a moderate stress of order 10^6Nm^{-2} induces an elastic strain of order 10^{-4}, to be compared with creep strains of order 10^{-2} in many applications. Thus elastic strain jumps are commonly neglected in comparison with the total creep. There is a primary decelerating creep ($\ddot{e} < 0$), a secondary or approximately steady creep around the stationary point $\ddot{e} = 0$ at $t_m(\bar{\sigma})$, $e_m(\bar{\sigma})$, then an accelerating tertiary creep ($\ddot{e} > 0$), shown in Fig. 1a. The dashed line represents a possible long-time asymptotic behaviour in which $e \to 1$ (specimen length squeezed to zero), which corresponds to $\sigma \to 0$ and hence a long-time steady viscous response $r_E(\sigma) \to 0$ if maintained zero stress σ implies zero strain-rate r. The viscous fluid response normally assumed in glaciology adopts the minimum strain-rate $r_m(\sigma)$ which occurs at time $t_m(\bar{\sigma})$, though strictly requires the long-time response $r_E(\sigma)$. Mellor remarks that $r_E(\sigma)$, $r_m(\sigma)$, may not be too different at low stress, though laboratory time scales are too short to reach any firm conclusion. The asymptotic result (2.5) is not crucial to the model correlation over time scales of interest, and it is the shape of response shown by the solid lines which is relevant. Figure 1b shows the associated strain-rate $\dot{e}$, with minimum $\dot{e}_m(\bar{\sigma})$ at time $t_m(\bar{\sigma})$.

The typical stress response at constant displacement-rate (constant $\dot{e}$) is shown as a stress-strain curve in Fig. 2, where $e = \dot{e}t$. The maximum stress $\bar{\sigma}_M(\dot{e})$, which also has the unfortunate description failure stress even though no failure of the material occurs, occurs at strain $e_M(\dot{e})$ and time $t_M(\dot{e})$. Again the dashed line represents a possible asymptotic behaviour. Mellor suggests that both e_m and e_M are approximately 0·01 over a wide range of stress and strain-rate respectively, and that there are indications that the maximum stress $\bar{\sigma}_M(\dot{e})$ is the constant stress $\bar{\sigma}$ required to produce a minimum strain-rate $\dot{e}_m(\bar{\sigma}) = \dot{e}$. That is

$$\bar{\sigma}_M[\dot{e}_m(\bar{\sigma})] = \bar{\sigma} \quad \text{and} \quad \dot{e}_m[\bar{\sigma}_M(\dot{e})] = \dot{e}. \tag{2.5}$$

These features are not crucial to the model construction.

With the alternative constant stress (constant σ) and constant strain-rate (constant r) interpretations, the typical strain-rate and stress responses in time are shown in Figs 3 and 4 respectively. The long-time response is shown as an asymptotic strain-rate $r_E(\sigma)$ and asymptotic stress $\sigma_E(r)$, with the case $r_e(\sigma) < r_o(\sigma)$, where $r_o(\sigma)$ is the initial strain-rate, illustrated. Model construction for the case $r_o(\sigma) > r_e(\sigma)$ would be similar. For either interpretation, the families of curves for different constant $\bar{\sigma}$ or different constant σ are non-linear in $\bar{\sigma}$, σ respectively, as illustrated by the conventional glaciology viscous law $r_m(\sigma)$ represented by a power law or polynomial. Similarly the families for different constant $\dot{e}$ or r are non-linear in $\dot{e}$ and r respectively. Accurate details of these families of responses have not yet been established, but their accepted shapes dictate minimal forms for compatible differential relations. In particular, the non-monotonic response, decelerating creep followed by accelerating creep at constant σ, has implications, and has not, to my knowledge, arisen in other branches of rheology.

3. Viscoelastic fluid relation for uni-axial response

A fluid relation is independent of the strain from a fixed reference configuration, and depends only on strain rate relative to the current configuration. In particular a viscous fluid expresses σ as a function of r, and a viscoelastic fluid of differential type relates σ, r, and their material time derivatives. The conventional definition does not include stress time derivatives, but it is shown that such terms are essential to reproduce the response in Fig. 4. First note that a viscous fluid relation in which σ is a function of r, or vice-versa, gives the responses constant r to constant σ and constant σ to constant r, in conflict with Figs 3 and 4. Examination of the constant σ response alone, Fig. 3 (M) showed that a term in $\dot{r}$ (at least) is needed so that the constant σ response is described by a differential equation for r with non-constant solution. Similarly, non-constant σ for constant r requires a term in $\dot{\sigma}$ (at least) (MS). Construction of a frame-indifferent tensor relation between stress, stress-rate, strain-rate, strain-acceleration, shown later, leads to a uni-axial stress relation

$$\tfrac{2}{3}\hat{\psi}_1\sigma + \tfrac{2}{3}\hat{\psi}_3\ (\dot{\sigma} - r\sigma) = \hat{\phi}_1 r - \tfrac{1}{2}\hat{\phi}_2 r^2 + \hat{\phi}_3\dot{r}\ , \tag{3.1}$$

where the tensor response coefficients $\hat{\psi}_1, \hat{\psi}_3, \hat{\phi}_1, \hat{\phi}_2, \hat{\phi}_3$ are here functions of σ, r, and their derivatives.

Now consider the response shape Fig. 3 for constant σ. In the strain-rate range $r_m(\sigma) < r < r_e(\sigma)$ there are two values of $\dot{r}$ at each r, so the differential equation for $r(t)$ must yield a two-branch solution:

$$\begin{aligned} \dot{r}_-(t) &= R_-(r,\sigma) \le 0\ , \quad r_o \ge r \ge r_m\ , \\ \dot{r}_+(t) &= R_+(r,\sigma) \ge 0\ , \quad r_m \le r < r_e\ , \end{aligned} \tag{3.2}$$

where

$$R_-(r_m,\sigma) = R_+(r_m,\sigma) = 0\ ,\ R_+ \to 0 \ \text{ as } \ r \to r_e\ . \tag{3.3}$$

The family of data curves gives the functions $R_-(r,\sigma)$ and $R_+(r,\sigma)$. It is convenient to make the definition

$$R_+(r,\sigma) \equiv 0\ , \quad r_e \le r \le r_o\ , \tag{3.4}$$

so that both $R_-(r,\sigma)$ and $R_+(r,\sigma)$ are defined in a common domain $r_o \ge r \ge r_m$, though the branch $\dot{r}_+(t)$ does not exist in the extension. The appropriate differential equation for $r(t)$ therefore has the form

x

$$\dot{r}^2 + f(r,\sigma)\dot{r} = F(r,\sigma)\ , \quad r(0) = r_o(\sigma)\ , \tag{3.5}$$

where

$$2\begin{pmatrix}\dot{r}_-\\ \dot{r}_+\end{pmatrix} = -f \mp (f^2 + 4F)^{\frac{1}{2}} \tag{3.6}$$

with

$$F \geq 0, \quad f \geq 0 \Rightarrow \dot{r}_- \leq 0, \quad \dot{r}_+ \geq 0 . \tag{3.7}$$

Thus

$$f = -(R_+ + R_-), \quad (f^2 + 4F)^{\frac{1}{2}} = R_+ - R_- , \quad r_m \leq r \leq r_O, \tag{3.8}$$

which determine $f(r,\sigma)$, $F(r,\sigma)$, with

$$F \equiv 0 , \quad r_e \leq r \leq r_o \quad . \tag{3.8a}$$

Analogous relations hold for the case $r_o < r_e$. The functions $f(r,\sigma)$, $F(r,\sigma)$, are related to the response coefficients in (3.1).

A similar analysis of the response shape Fig. 4 for constant r shows that the differential equation for $\sigma(t)$ must yield a two-branch solution

$$\begin{aligned} \dot{\sigma}_+(t) &= \Sigma_+(r,\sigma) \geq 0 , \quad && 0 \leq \sigma \leq \sigma_M , \\ \dot{\sigma}_-(t) &= \Sigma_-(r,\sigma) \leq 0, \quad && \sigma_M \geq \sigma > \sigma_E , \end{aligned} \tag{3.9}$$

where

$$\Sigma_+(r,\sigma_M) = \Sigma_-(r, \sigma_M) = 0 , \quad \Sigma_- \to 0 \quad \text{as} \quad \sigma \to \sigma_E . \tag{3.10}$$

The functions $\Sigma_+(r,\sigma)$, $\Sigma_-(r,\sigma)$, with the extension

$$\Sigma_-(r,\sigma) \equiv 0 , \quad \sigma_E \geq \sigma \geq 0 , \tag{3.11}$$

are given by the data curves. The appropriate differential equation for $\sigma(t)$ has the form

$$\dot{\sigma}^2 - g(r,\sigma)\dot{\sigma} = G(r,\sigma), \quad \sigma(0) = 0 \tag{3.12}$$

where

$$2\begin{pmatrix}\dot{\sigma}_+ \\ \dot{\sigma}_-\end{pmatrix} = g \pm (g^2 + 4G)^{\frac{1}{2}} \tag{3.13}$$

with

$$G \geq 0, \quad g \geq 0 \Rightarrow \dot{\sigma}_+ \geq 0, \quad \dot{\sigma}_- \leq 0. \tag{3.14}$$

Thus

$$g = \Sigma_+ + \Sigma_-, \quad (g^2 + 4G)^{\frac{1}{2}} = \Sigma_+ - \Sigma_-, \quad 0 \leq \sigma \leq \sigma_M, \tag{3.15}$$

which determine $g(r,\sigma)$, $G(r,\sigma)$, with

$$G \equiv 0, \quad 0 \leq \sigma \leq \sigma_E . \tag{3.16}$$

The functions $g(r,\sigma)$, $G(r,\sigma)$, are related to the response coefficients in (3.1).

Now there are four relations in (3.8) and (3.15) to determine the four functions f, F, g, G, in terms of the data functions R_-, R_+, Σ_+, Σ_-. However, the differential equation (3.5) must follow from (3.1) when $\dot{\sigma} = 0$, and (3.12) must follow from (3.1) when $\dot{r} = 0$, for the same set of response coefficients ψ_1, ψ_3, ϕ_1, ϕ_2, ϕ_3. The forms (3.5), (3.12) can be derived from (3.1) by a variety of response coefficient assumptions, but for a series of constructions (not exhaustive) (MS), in each case

$$F \equiv G . \tag{3.17}$$

Thus, for these models, only three of the data functions R_-, R_+, Σ_+, Σ_-, can be independent, so Mellor's conjecture that constant strain-rate response is determined by constant stress response (or vice-versa) is partially, but not fully, realised. For example, the three independent functions f, F, g can be determined from the full constant stress response R_- and R_+ together with the primary constant strain-rate stage Σ_+, leaving the stress relaxation Σ_- predicted. If the four data functions are not compatible with the identity (3.17), then an alternative construction and possibly a more general relation than (3.1) are required. Note that the identity (3.17) and various F and G relations also imply the inverse relation analogous to (2.5) and the inverse relation for the long-time response:

$$r_m[\sigma_M(r)] = r\,, \qquad r_e[\sigma_E(r)] = r\,. \tag{3.18}$$

Given the three independent data functions, only three response coefficients in (3.1) can be determined of the four required after normalising (dividing throughout by $\hat{\psi}_1$ or $\hat{\psi}_3$ for example, if not zero). However, in uni-axial response, the $\hat{\phi}_1$ and $\hat{\phi}_2$ terms form a single composite term which can be separated only by multi-axial response. But the different normalisations and constructions lead to different models for the same data, so the data does not determine a unique model of the form (3.1). Again it is multi-axial response which will distinguish the merits of the different models.

4. Viscoelastic solid relation for uni-axial response

A solid material relation is obtained by including dependence on the strain e from the reference configuration, and it is convenient to describe the response in terms of the nominal stress

$\bar{\sigma}$. The non-uniform $\bar{\sigma}(e)$ shown in Fig. 2 for constant $\dot{e}$ again implies that a stress-rate term $\dot{\bar{\sigma}}$ is required in a differential relation. At constant $\bar{\sigma}$, the non-uniform $e(t)$ and $\dot{e}(t)$ shown in Fig. 1 imply that a differential equation is required for $e(t)$, and this is achieved by terms in strain e and strain-rate $\dot{e}$. Thus the strain term replaces the strain-acceleration term necessary in a fluid relation where strain dependence is excluded. The uni-axial relation therefore involves terms in $\bar{\sigma}$, $\dot{\bar{\sigma}}$, e, $\dot{e}$ (SM). Construction of a frame-indifferent tensor relation between stress, stress-rate, strain, and strain-rate, shown later, leads to a uni-axial stress relation

$$(1-e)^3\bar{\sigma} + \hat{\psi}(1-e)^2[(1-e)\dot{\bar{\sigma}} - 2\dot{e}\bar{\sigma}] = \tfrac{3}{2}\hat{\phi}(1-e)\dot{e} + \hat{\omega}e\,, \qquad (4.1)$$

where the three response coefficients $\hat{\psi}$, $\hat{\phi}$, $\hat{\omega}$, are functions of $\bar{\sigma}$ and e. That is, the tensor response coefficients have been limited to dependence on stress and strain invariants, and not on their rates, since this leaves more than enough flexibility to match the responses shown in Figs 1 and 2. Further, it has been assumed that the stress and stress-rate tensors enter only in a linear combination, and that the strain and strain-rate tensors enter as separate terms. The form (4.1) is derived from an isotropic law for the reference configuration, and the $\hat{\omega}$, $\hat{\phi}$ terms are each composites of the uni-axial restrictions of two tensor terms. There are five response coefficients in the tensor relation, and the normalisation shown in (4.1) implies that the stress tensor must be present.

Since the differential fluid relation has explicit dependence on the strain-acceleration $\dot{r}$, the non-monotonic $r(t)$, Fig. 3, at constant σ gives two functions R_- and R_+ for $\dot{r}(\sigma,r)$. Here there is no explicit dependence on $\ddot{e}$ so we can expect only one independent data function from the constant $\bar{\sigma}$ response. Also, for the response to constant displacement-rate, constant $\dot{e}$, the non-monotonic $\bar{\sigma}(e)$ response, Fig. 2, does not imply that $d\bar{\sigma}/de$ is a double-valued function of $(\bar{\sigma},e)$, since at each repeated value of $\bar{\sigma}$, e is distinct. Since the uni-axial relation involves three reponse coefficients $\hat{\psi}$, $\hat{\phi}$, $\hat{\omega}$, two other basic tests have been analysed to see whether their response data, if obtained, could provide further indpendent relations between the response coefficients. It is found that they provide no independent relations, though one or both may be useful alternative experimental tests. It is then shown that no uni-axial stress loading can provide a third independent relation, so that the complete uni-axial model requires multi-axial tests. Reduced models which can be determined by two uni-axial tests are examined, along with their main features so that a trial model may be adopted in advance of multi-axial data.

It is convenient to introduce the following explicit test terminology and abbreviations:-

CL: Constant load (constant nominal stress),

CLR: Constant load rate (constant nominal stress-rate),

CD: Constant end displacement (constant engineering strain),

CDR: Constant end displacement rate (constant engineering strain-rate).

Corresponding constant stress and constant strain-rate tests then refer to current configuration measures.

At constant $\bar{\sigma}$ (CL), (4.1) gives

$$\dot{e}[\tfrac{3}{2}\hat{\phi}(1-e) + 2\hat{\psi}\bar{\sigma}(1-e)^2] = (1-e)^3\bar{\sigma} - \hat{\omega}e \quad . \tag{4.2}$$

Now it is assumed that the strain-response $e(t)$, Fig. 1a, is monotonic, so that t can be expressed as a single-valued function of e for each $\bar{\sigma}$:

$$t = T(\bar{\sigma},e) \quad . \tag{4.3}$$

Then the strain-rate response $\dot{e}(t)$, Fig. 1b, can be expressed as a function of e for each $\bar{\sigma}$, thus

$$\text{CL:} \quad \dot{\bar{\sigma}} = 0 \,, \qquad \dot{e} = F(\bar{\sigma},e) \,, \qquad e(0) = e_e(\bar{\sigma}) \,, \tag{4.4}$$

where the data function $F(\bar{\sigma},e)$ is related to the response coefficients through (4.2) which gives

$$F(\bar{\sigma},e) = \frac{(1-e)^3\bar{\sigma} - \hat{\omega}e}{\frac{3}{2}\hat{\phi}(1-e) + 2\hat{\psi}\bar{\sigma}(1-e)^2} \quad . \tag{4.5}$$

This function F is not related to that used in the previous fluid model analysis.

At constant $\dot{e} = w$ (CDR), it is assumed that the family of response curves $\bar{\sigma}(e)$ for different w, Fig. 2, do not intersect; that is, at each e, $\bar{\sigma}$ increases with w, which is consistent with an increasing peak stress $\sigma_M(w)$ at constant e_M (independent of w). Then there is a monotonic $\bar{\sigma}(w)$ relation at each e so that w can be expressed as a single-valued function of $\bar{\sigma}$ for each e:

$$w = W(\bar{\sigma},e) . \tag{4.6}$$

From the family $\bar{\sigma}(e)$ for different w, a generalised Young's modulus can be defined and expressed as a function $E(\bar{\sigma},e)$ by eliminating w through (4.6), thus

$$\text{CDR:}\quad \dot{e} = w\,, \qquad \bar{\sigma} = G(w,e)\,, \quad E(\bar{\sigma},e) = \left.\frac{\partial G}{\partial e}\right|_{w} . \tag{4.7}$$

That is, $E(\bar{\sigma},e)$ is the stress-strain gradient on the constant strain-rate response. The function pair $W(\bar{\sigma},e)$, $E(\bar{\sigma},e)$ of the CDR response family are related to the response coefficients by setting $\dot{e} = w$ in (4.1), and using (4.~~0~~5) and (4.6) gives

$$E(\bar{\sigma},e) = \hat{E}(\bar{\sigma},e)\left\{1 - \frac{F(\bar{\sigma},e)}{W(\bar{\sigma},e)}\right\}\,, \tag{4.8}$$

where

$$\hat{E}(\bar{\sigma},e) = \frac{3\hat{\phi}}{2(1-e)^2\hat{\psi}} + \frac{2\bar{\sigma}}{1-e} . \tag{4.9}$$

Thus the CL and CDR tests yield two independent relations (4.5) and (4.9) to determine three response coefficients $\hat{\psi}(\bar{\sigma},e)$, $\hat{\phi}(\bar{\sigma},e)$, $\hat{\omega}(\bar{\sigma},e)$. The factor $\hat{E}(\bar{\sigma},e)$ has a natural rôle in the jump relation derived later.

Now consider the alternative CLR and CD tests for which there are no established typical response curves. At constant load-rate $\dot{\bar{\sigma}} = q$ (CLR), assume that there are a family of monotonic non-intersecting strain responses $e(t)$ for different q, as illustrated in Fig. 5a. It is supposed (in accord with general observation) that the time to reach a given strain e increases as the stress-rate q increases, then the corresponding curves for e as a function of $\bar{\sigma} = qt$ fan out, as shown in Fig. 5b.

Thus, at each $\bar{\sigma}$, there is a monotonic decreasing $e(q)$ relation which can be inverted,

$$\text{CLR:} \quad \dot{\bar{\sigma}} = q \, , \quad e = e^*(\bar{\sigma}, q) \, , \quad q = Q(\bar{\sigma}, e) \, , \tag{4.10}$$

with the derivative signs

$$\frac{\partial e^*}{\partial \bar{\sigma}} > 0 \, , \quad \frac{\partial e^*}{\partial q} < 0 \, , \quad \frac{\partial Q}{\partial \bar{\sigma}} > 0 \, , \quad \frac{\partial Q}{\partial e} < 0 \quad . \tag{4.11}$$

In turn, the strain-rate on $\dot{\bar{\sigma}} = q$ can be expressed in terms of $\bar{\sigma}, e$, by eliminating q through (4.10), thus

$$\dot{e} = q \frac{\partial e^*}{\partial \bar{\sigma}} = D(\bar{\sigma}, e) \, , \quad > 0 \tag{4.12}$$

by (4.11). The function pair $Q(\bar{\sigma}, e)$, $D(\bar{\sigma}, e)$ are related to the response coefficients by setting $\dot{\bar{\sigma}} = q$ in (4.1), thus

$$D(\bar{\sigma}, e) = \frac{(1-e)^3 \bar{\sigma} - \hat{\omega} e + (1-e)^3 \hat{\psi} Q(\bar{\sigma}, e)}{\frac{3}{2}\hat{\phi}(1-e) + 2\hat{\psi}\bar{\sigma}(1-e)^2} \quad . \tag{4.13}$$

Since strain increases monotonically at constant stress, it is reasonable to suppose that stress decreases monotonically from the initial elastic stress jump at constant engineering strain (CD). Thus, at each $\bar{e}$, t can be expressed as a single-valued function of $\bar{\sigma}$,

$$t = T^*(\bar{\sigma}, e) \, , \qquad \bar{\sigma} \le e_e^{-1}(e) \, , \tag{4.14}$$

and the stress-rate expressed as a function of $\bar{\sigma}, e$:

$$\text{CD:} \quad \dot{e} = 0 \, , \quad \dot{\bar{\sigma}} = - L(\bar{\sigma}, e) < 0 \, , \quad \bar{\sigma} \le e_e^{-1}(e) . \tag{4.15}$$

By (4.1),

$$L(\bar{\sigma},e) = \frac{(1-e)^3\,\bar{\sigma} - \hat{\omega}e}{(1-e)^3\,\hat{\psi}} \quad . \tag{4.16}$$

Table 1 summarises the data functions, and their relation to response coefficients, determined by the four tests discussed above. Note that the relations (4.8), (4.9) in CDR are a single

Table 1 Test functions

Test	Data functions	Relation
CL	$F(\bar{\sigma},e)$	(4.5)
CDR	$W(\bar{\sigma},e)$, $E(\bar{\sigma},e)$	(4.8),(4.9)
CLR	$Q(\bar{\sigma},e)$, $D(\bar{\sigma},e)$	(4.13)
CD	$L(\bar{\sigma},e)$	(4.16)

relation for the response coefficients in terms of the data-functions W and E when F is eliminated by (4.5). Thus the four tests give four relations for the three response coefficients. However, only two relations can be independent because (4.1) can be expressed in a form involving only two combinations of the coefficients, namely

$$\dot{\bar{\sigma}} = \hat{E}(\bar{\sigma},e)\dot{e} - L(\bar{\sigma},e) \quad , \tag{4.17}$$

where the definitions (4.9), (4.16), are used. That is, the responses described by any of the four tests in Table 1, or in fact by any prescribed loading history, can only give relations involving the combinations $\hat{E}(\bar{\sigma},e)$ and $L(\bar{\sigma},e)$. Results of more than two tests therefore provide only consistency checks for the adopted model.

Table 2 shows the uni-axial relation (4.1) expressed in terms of measured functions from each of the six possible pairs of tests described above. Note that $\hat{E}$ is a combination of

Table 2 Uni-axial relation expressions

Test pair	Relation
CL and CDR	$\dot{\bar{\sigma}} = \hat{E}(\dot{e}-F)$
CL and CLR	$(D-F)\dot{\bar{\sigma}} = Q(\dot{e}-F)$
CDR and CLR	$(W-D)(\dot{\bar{\sigma}}-Q) = (EW-Q)(\dot{e}-D)$
CL and CD	$F\dot{\bar{\sigma}} = L(\dot{e}-F)$
CDR and CD	$W(\dot{\bar{\sigma}}+L) = (EW+L)\dot{e}$
CLR and CD	$D(\dot{\bar{\sigma}}+L) = (L+Q)\dot{e}$

F, W, E (4.8) given by CL and CDR. The implication of (4.17) and the Table 2 expressions is a sequence of identities between the sets of data functions from different pairs of tests, given directly by the definitions (4.5), (4.8), (4.9), (4.13), (4.16). The most simple direct expression is that from CL and CD which are each described by a single function, F and L respectively. Also the CL and CDR expression using the combination function $\hat{E}$ and F is convenient. The form (4.17) requires functions from three tests, but with the identity $L = \hat{E}F$ is simply the CL and CDR expression.

It is possible to obtain the simple form (4.17) involving only two coefficients $E(\bar{\sigma},e)$, $L(\bar{\sigma},e)$ because the scalar uni-axial relation (4.1) allows division by a function of $(\bar{\sigma},e)$. In the tensor form constructed later, such division is not possible, so the extra cofficient, and separation of the composite coefficients $\hat{\phi}$ and $\hat{\omega}$, results in five independent coefficients. These are necessary to represent the distinct directional features of the different tensor terms, and in principle must be determinate by an appropriate set of multi-axial tests.

Now consider how the strain response to an applied stress jump - a jump relation - can be inferred from the differential relation (4.1) or its equivalent form (4.17). A reasonable proposition is that the jump relation is the limit of solutions of the differential relation in which the stress change takes place over a decreasing sequence of time intervals. Consider a continuous stress change $\bar{\sigma}_o$ to $\bar{\sigma}$ in the time interval t_o to $t_o + \delta t$, with corresponding strain change e_o to e, then integrating (4.17)

$$\int_{t_o}^{t_o+\delta t} \dot{\bar{\sigma}} dt = \int_{t_o}^{t_o+\delta t} \hat{E}(\bar{\sigma}',e')\dot{e}'dt - \int_{t_o}^{t_o+\delta t} L(\bar{\sigma}',e')dt \ . \tag{4.18}$$

Assuming that $\bar{\sigma}'$, e' remain bounded, and L is a bounded function, the last term of (4.18) is of order δt as $\delta t \to 0$. Thus,

$$\bar{\sigma} - \bar{\sigma}_o = \int_{e_o}^{e} \hat{E}(\bar{\sigma}',e')de' + O(\delta t) \ , \tag{4.19}$$

where the argument $\bar{\sigma}'$ of $\hat{E}$ runs through the values $\bar{\sigma}_o$ to

$\bar{\sigma}$. Assume that in the limit $\delta t \to 0$

$$\bar{\sigma} = J(e, e_o, \bar{\sigma}_o), \quad \bar{\sigma}_o = J(e_o, e_o, \bar{\sigma}_o) \ , \tag{4.20}$$

which is the jump relation for arbitrary stress jump $\bar{\sigma}_o \to \bar{\sigma}$, then for bounded $\hat{E}$, (4.19) gives

$$\frac{\partial J}{\partial e} = \frac{d\bar{\sigma}}{de} = \hat{E}[J(e, e_o, \bar{\sigma}_o), e] = \hat{E}(\bar{\sigma}, e) \ , \tag{4.21}$$

showing that $E(\bar{\sigma}, e)$ is the stress-strain gradient at a point $(\bar{\sigma}, e)$ during a jump. The complete jump is given by integrating (4.21) with the initial condition $(4.20)_2$ at $e = e_o$, which is the integral (4.19) in the limit $\delta t \to 0$. An analogous tensor relation jump can also be constructed, and bounded integrals have implications for the response coefficients ϕ_1, ϕ_2 arising in the composite coefficient $\hat{\phi}$. Note that jump relation data can only provide information about the coefficient combination $\hat{E}$.

There remains the question of what stress-strain domains are covered by the different tests when only compressive stress $(\bar{\sigma} > 0)$ is applied. Since the response coefficients are, in general, functions of $(\bar{\sigma}, e)$, the relation (4.1) can describe uni-axial response only for stress-strain histories within their common domain of definition. In constant load tests over a load range $0 < \bar{\sigma} \le \bar{\sigma}_L$, taken to a strain limit e_ℓ, $F(\bar{\sigma}, e)$ is defined only between the initial elastic strain $e_e(\bar{\sigma})$ and the upper limit, thus

$$\text{CL: } F(\bar{\sigma}, e), \ 0 < \bar{\sigma} \le \bar{\sigma}_L \ , \quad e_e(\bar{\sigma}) \le e \le e_\ell, \ F(0, e) \equiv 0 \ . \tag{4.22}$$

For constant displacement tests over a strain range $0 < e < e_\ell$, at each constant e the stress is bounded above by the initial elastic stress jump $e_e^{-1}(e)$, (4.15), so

$$\text{CD: } L(\bar{\sigma},e), \quad e_e(\bar{\sigma}) < e \le e_\ell, \quad L(\bar{\sigma},0) \equiv 0 , \tag{4.23}$$

which has the same domain (4.22) as the CL test. In the constant displacement-rate test there is a limit stress-strain curve, of the form shown in Fig. 2, which depends on the constant strain-rate $\dot{e} = w$ value. The initial slope for any w is

$$\text{CDR: } \frac{d\bar{\sigma}}{de} = E(0,0) = \hat{E}(0,0) = E_o , \tag{4.24}$$

independent of w (a restriction of the model). By (4.21), the initial slope of the elastic jump relation $\bar{\sigma} = e_e^{-1}(e)$ is also E_o, and hence the limit curve of the CL (and CD) domain is tangent to that of the CDR test at (0,0). Since $e_e(\bar{\sigma})$ is monotonic, and the limit stress-strain curve of CDR, Fig. 2, has a turning point, the CDR domain is more restricted than the CL domain as $\bar{\sigma}$ increases. In the constant load-rate test, the limit stress-strain curve, Fig. 5b, depends on the maximum load-rate q. Its initial slope is

$$\text{CLR: } \frac{d\bar{\sigma}}{de} = \frac{Q(0,0)}{D(0,0)}, = \hat{E}(0,0)\left\{1 - \frac{F(0,0)}{D(0,0)}\right\} = E_o , \tag{4.25}$$

using the identities implied by Table 2, independent of q, which is the initial slope of CL, CD, and CDR limit curves. For $\bar{\sigma} > 0$, $Q/D < \hat{E}$ since F and D are positive, so the CLR limit curve, though monotonic, bounds a more restricted domain than the jump relation $e_e(\bar{\sigma})$. The CDR limit curve is the most restrictive.

At best, using the CL and CD tests for $\bar{\sigma} > 0$, there is an excluded domain $0 \leq e < e_e(\bar{\sigma})$ not entered by the loading histories of the four above tests. However, a state $\bar{\sigma} > 0$, $0 < e < e_e(\bar{\sigma})$, can be reached by unloading histories which involve some previous arbitrarily small tension $(\bar{\sigma} < 0)$; an illustration is presented in Report (M7).

5. Reduced viscoelastic solid relation

While multi-axial tests can, in principle, determine all the response coefficients of the tensor relation shown later, and hence describe the uni-axial response fully, in the absence of such extensive multi-axial data it is useful to construct a reduced model which is determined by uni-axial data. That is, a model which requires only two independent response coefficient combinations. This cannot, of course, determine the directional features of a tensor relation, but can be extrapolated in various ways to construct trial tensor relations.

First consider the removal of one term in the uni-axial relation (4.1) by setting $\hat{\phi}$, $\hat{\psi}$, $\hat{\omega}$, zero in turn; that is, eliminating the strain-rate tensor, stress-rate tensor, and strain tensor respectively. If $\hat{\phi} = 0$, by (4.9)

$$\hat{E}(\bar{\sigma},e) = \frac{2\bar{\sigma}}{1-e} = O(\bar{\sigma}), \quad \hat{E}(0,0) = 0. \tag{5.1}$$

Now $\hat{E}$ is a modulus, in particular defining the jump stress-strain gradient (4.21), so that $\hat{E} >> \bar{\sigma}$ for $e << 1$ which contradicts (5.1). Hence removal of the strain-rate tensor term is not physically acceptable. If $\hat{\psi} = 0$, then $\hat{E}$ is infinite and there is no strain jump for an applied stress jump, which is an acceptable approximation since strain-jumps are small compared with the usual creep strains. Bounded E then implies W = F. F and D are bounded, and identical, (4.5) and (4.13), and Q is bounded and non-zero by definition. So from the CL and CLR expression in Table 2, $\dot{e} = F(\bar{\sigma},e)$ for all $\bar{\sigma}(t)$, not just $\bar{\sigma}$ = constant. However, in a constant displacement test, setting $\dot{e} = 0$ in (4.1),

$$\text{CD:} \quad \dot{e} = 0 \,, \qquad (1-e)^3\,\bar{\sigma} = \hat{\omega}(\bar{\sigma},e)\,e \,, \tag{5.2}$$

which implies that $\bar{\sigma}$ = constant (a solution of the implicit equation for $\bar{\sigma}$ at constant e), incompatible with non-zero L in (4.15). Hence removal of the stress-rate tensor is not physically acceptable.

If $\hat{\omega} = 0$, there is no dependence on the strain tensor, but dependence on strain invariants remains in the arguments of $\hat{\phi}$ and $\hat{\psi}$, and induced anisotropy still occurs (SM). Now by (4.5),

$$F(0,e) = 0 \,, \tag{5.3}$$

so that on complete unloading from any stress-strain state there is the elastic strain decrease but no subsequent creep relaxation

($\dot{e} = 0$). A linear viscoelastic solid which exhibits decreasing strain-rate in time at constant stress, because of the superposition property always relaxes ($\dot{e} < 0$) on full or partial unloading. On partial unloading, $\bar{\sigma}$ maintained at a reduced positive level, $\dot{e} = F(\bar{\sigma},e) > 0$ when $\hat{\omega} = 0$, so creep continues at a reduced rate. A non-vanishing $\hat{\omega}$ is essential to describe creep relaxation on full or partial unloading. If a non-relaxing model $\hat{\omega} = 0$ is an acceptable approximation, then the remaining coefficients $\hat{\phi}$, $\hat{\psi}$ are determined by any pair of tests in Table 2. For example, from CL and CDR

$$\hat{\psi} = \frac{\bar{\sigma}}{F\hat{E}} , \quad \hat{\phi} = \frac{2\bar{\sigma}(1-e)}{3F\hat{E}} \left\{ (1-e)\hat{E} - 2\bar{\sigma} \right\} . \tag{5.4}$$

When $\hat{\omega} \neq 0$, relaxation on complete unloading from a state in which $\dot{e} = F(\bar{\sigma},e) > 0$ requires $\dot{e} = F(0, e-e_j) < 0$, where e_j is the small elastic strain decrease, trivially satisfied if $\hat{\omega} > 0$. Relaxation after a small stress decrease, analogous to the linear model, would require significant increase in $\hat{\omega}$ for the small stress, small strain decrease. Unloading data can determine the importance of the strain tensor term with coefficient $\hat{\omega}$.

An alternative reduction of the model is obtained by restricting dependence of the response coefficients $\hat{\psi}$, $\hat{\phi}$, $\hat{\omega}$, on $\bar{\sigma}$ and e. The uni-axial relation (4.1) is the restriction of a tensor relation with unit stress tensor coefficient. If the response coefficients have general dependence on the stress and strain invariants, then the tensor relation may be expressed with unit stress-rate tensor coefficient without loss of generality. However, making analogous restrictions on the dependence of the two sets of coefficients leads to different models, so both will be analysed. The uni-axial relation for the second normalisation is

$$(1-e)^3 \dot{\bar{\sigma}} - 2(1-e)^2 \bar{\sigma} \dot{e} + (1-e)^3 \hat{\psi}^* \bar{\sigma} = \frac{3}{2}\hat{\phi}^*(1-e)\dot{e} + \hat{\omega}^* e \tag{5.5}$$

where the three response coefficients $\hat{\psi}^*$, $\hat{\phi}^*$, $\hat{\omega}^*$, are functions of $\bar{\sigma}$ and e in general, and the $\hat{\phi}^*$, $\hat{\omega}^*$ terms are each composites of two tensor terms. Both (4.1) and (5.5) can be written in the common form associated with CL and CDR responses, Table 2,

$$\dot{\bar{\sigma}} = \hat{E}(\dot{e}-F) , \tag{5.6}$$

with F, $\hat{E}$ defined by (4.5), (4.9) for (4.1), and for (5.5),

$$\hat{E} = \frac{3\hat{\phi}^*}{2(1-e)^2} + \frac{2\bar{\sigma}}{1-e} , \quad F = \left\{ \hat{\psi}^* \bar{\sigma} - \frac{\hat{\omega}^* e}{(1-e)^3} \right\} \Big/ \hat{E} . \tag{5.7}$$

Consider a requirement that creep relaxation takes place on complete unloading from a state $(\bar{\sigma}_1, e_1)$ at time t_1. Thus

$$t > t_1 : \bar{\sigma} \doteq 0 , \quad \dot{e} = F(0,e) = -f(e) < 0, \quad e(t_1^+) = e_1^+ , \tag{5.8}$$

where $e_1 - e_1^+$ is the elastic strain decrease. While the strain-rate $-f(e)$ is a function only of current strain, the initial value e_1^+ depends on the loading history to reach the state $(\bar{\sigma}_1, e_1)$ and the elastic jump from e_1 to e_1^+. For (4.5), (4.9), and (5.7) respectively,

$$f(e) = \frac{\hat{\omega}(0,e)e}{\frac{3}{2}(1-e)\hat{\phi}(0,e)} = \frac{\hat{\omega}(0,e)e}{(1-e)^3\,\hat{E}(0,e)\hat{\psi}(0,e)}, \tag{5.9}$$

$$f(e) = \frac{\hat{\omega}^*(0,e)e}{\frac{3}{2}(1-e)\hat{\phi}^*(0,e)} = \frac{\hat{\omega}^*(0,e)e}{(1-e)^3\,\hat{E}(0,e)}. \tag{5.10}$$

Thus the relaxation function $f(e)$ determines the ratios $\hat{\omega}(0,e)/\hat{\phi}(0,e)$ and $\hat{\omega}^*(0,e)/\hat{\phi}^*(0,e)$ or $\hat{\omega}^*(0,e)/\hat{E}(0,e)$ evaluated at zero stress only. In practice the function $F(0,e)$ cannot be determined by an initial zero load test, so a relaxation test completes the determination of $F(\bar{\sigma},e)$ if e_1^+ and the final strain encompass the required range.

Suppose now that $\hat{\omega}/\hat{\phi}$ is independent of $\bar{\sigma}$, hence equal to $\hat{\omega}(0,e)/\hat{\phi}(0,e)$ determined by $f(e)$, which therefore provides a third relation on the response coefficients. Then by (5.9), (4.5) (4.9),

$$\hat{\omega}e = \tfrac{3}{2}\hat{\phi}(1-e)f, \qquad \hat{\psi} = \frac{\frac{3}{2}\hat{\phi}}{(1-e)^2\hat{E} - 2(1-e)\bar{\sigma}},$$

$$\frac{3}{2}\hat{\phi} = \frac{(1-e)^2\bar{\sigma}\{(1-e)\hat{E} - 2\bar{\sigma}\}}{2\bar{\sigma}F + (F+f)\{(1-e)\hat{E} - 2\bar{\sigma}\}}. \tag{5.11}$$

Similarly, $\hat{\omega}^*/\hat{\phi}^*$ is then independent of $\bar{\sigma}$, and direct from (5.10) and (5.7), or by the transformations

$$\hat{\psi}^* = \frac{1}{\check{\psi}} , \qquad \hat{\phi}^* = \frac{\hat{\phi}}{\check{\psi}} , \qquad \hat{\omega}^* = \frac{\hat{\omega}}{\check{\psi}} , \tag{5.12}$$

used to obtain (5.5) from (4.1),

$$\frac{3}{2}\hat{\phi}^* = (1-e)\{(1-e)\hat{E} - 2\bar{\sigma}\} , \quad \omega^* e = (1-e)^2 \{(1-e)\hat{E} - 2\bar{\sigma}\} f ,$$

$$\psi^* = \frac{2\bar{\sigma}F + (F+f)\{(1-e)\hat{E} - 2\bar{\sigma}\}}{(1-e)\bar{\sigma}} . \tag{5.13}$$

Since elastic strains associated with a stress jump $\bar{\sigma}$ are infinitesimal,

$$\frac{\bar{\sigma}}{\check{E}} << 1 , \tag{5.14}$$

then neglecting $\bar{\sigma}/\hat{E}$ compared to unity gives the approximations of (4.9) and $(5.7)_1$,

$$\hat{E} = \frac{3\hat{\phi}}{2(1-e)^2 \check{\psi}} = \frac{3\hat{\phi}^*}{2(1-e)^2} , \tag{5.15}$$

which imply

$$\hat{\phi} >> \bar{\sigma}\hat{\psi} , \qquad \hat{\phi}^* >> \bar{\sigma} , \tag{5.16}$$

and in turn the approximate expressions of (4.5) and $(5.7)_2$,

$$F = \frac{(1-e)^3\,\bar{\sigma} - \hat{\omega}e}{\frac{3}{2}\hat{\phi}(1-e)} = \frac{(1-e)^3\,\bar{\sigma}\hat{\psi}^* - \hat{\omega}^*e}{\frac{3}{2}\hat{\phi}^*(1-e)} \quad . \tag{5.17}$$

The corresponding approximations of (5.11) and (5.13) are

$$\frac{3}{2}\hat{\phi} = \frac{(1-e)^2\,\bar{\sigma}}{F+f} \;, \quad \hat{\psi} = \frac{\bar{\sigma}}{(F+f)\hat{E}} \;, \quad \hat{\omega}e = \frac{(1-e)^3\,\bar{\sigma}f}{F+f} \;, \tag{5.18}$$

$$\frac{3}{2}\hat{\phi}^* = (1-e)^2\,\hat{E} \;, \quad \hat{\psi}^* = \frac{(F+f)\hat{E}}{\bar{\sigma}} \;, \quad \hat{\omega}^*e = (1-e)^3\,\hat{E}f \;. \tag{5.19}$$

While total strain e may be small, it is not necessarily negligible compared to unity, and in any case there is no further simplification of the forms (5.18), (5.19), by neglecting e compared to unity. Note that $(F+f)$ must approach zero like $\bar{\sigma}$ or faster as $\bar{\sigma} \to 0$ for bounded $\hat{\psi}^*$.

The three response coefficients in both relations (4.1) and (5.5) are determined by the two types of loading response (CL and CDR) and the response on complete unloading, provided that $\hat{\omega}/\hat{\phi}$ (and $\hat{\omega}^*/\hat{\phi}^*$) are independent of $\bar{\sigma}$. Until this is shown to be incompatible with multi-axial load data it is a useful trial model, with the forms (4.1) and (5.5) still equivalent. Now consider the further simplifications (Report M8) of separable dependence in $\hat{\omega}$, $\hat{\phi}$, and $\hat{\psi}^*$, $\hat{\phi}^*$, respectively, compatible with the ratios independent of $\bar{\sigma}$. Thus

$$\hat{\phi} = \phi_\sigma(\bar{\sigma})\phi_e(e), \quad \hat{\omega}e = \tfrac{3}{2}\phi_\sigma(\bar{\sigma})\phi_e(e)(1-e)f(e)\ , \tag{5.20}$$

$$\hat{\phi}^* = \phi^*_\sigma(\bar{\sigma})\phi_e{}^*(e), \ \omega^*e = \frac{3}{2}\ \phi^*_\sigma(\bar{\sigma})\phi^*_e(e)(1-e)f(e)\ . \tag{5.21}$$

The corresponding jump gradients $\hat{E}$ from the approximate expressions (5.15) are

$$\hat{E} = \frac{3\phi_\sigma(\bar{\sigma})\phi_e(e)}{2(1-e)^2\,\hat{\psi}(\bar{\sigma},e)} = \frac{3\phi^*_\sigma(\bar{\sigma})\phi^*_e(e)}{2(1-e)^2}\ , \tag{5.22}$$

of which the first expression is not separable when $\hat{\psi}$ is unrestricted, while the second expression is separable. That is, the two models (4.1) and (5.5) are distinct, and this feature of $\hat{E}$ may decide which provides the more plausible approximation. It has been shown by model examples (SM) that variation of $\hat{E}$ with e has little effect on the CDR response, and so a simplified separable form $\hat{E} = \hat{E}(\bar{\sigma})$ may be an adequate approximation, supporting the expression $(5.22)_2$ and relation (5.5). From (5.17),

$$F + f(e) = \frac{(1-e)^2}{\frac{3}{2}\phi_e(e)} \cdot \frac{\bar{\sigma}}{\phi_\sigma(\bar{\sigma})} = \frac{(1-e)^2\ \bar{\sigma}\ \hat{\psi}^*(\bar{\sigma},e)}{\frac{3}{2}\ \phi^*_e(e)\phi^*_\sigma(\bar{\sigma})}\ , \tag{5.23}$$

requiring $F + f(e)$ to be separable for the relation (4.1), but not restricted for the relation (5.5) when $\hat{\psi}^*$ is unrestricted. Hence the relation (4.1) with separable coefficients (5.20) requires that $F(\bar{\sigma},e) + f(e)$ is separable, while the relation (5.5) with separable coefficients (5.21) requires that $\hat{E}(\bar{\sigma},e)$

is separable.

Further simplifications are given by the special cases of (5.20) and (5.21) when all $\bar{\sigma}$ dependence is eliminated:

$$\hat{\phi} = \phi_e(e), \qquad \hat{\omega}e = \tfrac{3}{2}\phi_e(e)(1-e)f(e) \;, \tag{5.24}$$

$$\hat{\phi}^* = \phi_e^*(e), \qquad \omega^* e = \tfrac{3}{2}\phi_e^*(e)(1-e)f(e) \;. \tag{5.25}$$

With (5.24), $\hat{E}$ is unrestricted, $(5.22)_1$, but $F + f(e)$ is linear in $\bar{\sigma}$, $(5.23)_1$, while with (5.25), $\hat{E} = \hat{E}(e)$, $(5.22)_2$, but F is unrestricted. In particular, if the approximation $\hat{E}$ = constant = E_o is adopted, then

$$\tfrac{3}{2}\hat{\phi}^* = E_o(1-e)^2 \;, \quad \hat{\omega}^* e = E_o(1-e)^3 f(e), \quad \hat{\psi}^* = \frac{E_o[\,F(\bar{\sigma},e)+f(e)\,]}{\bar{\sigma}}, \tag{5.26}$$

which, with the relation (5.5), may be a useful simple model in view of the small influence of $\hat{E}(e)$ variation in CDR response illustrations (SM).

Finally, let us examine the properties of the strain-rate $-f(e)$ on complete unloading, defined by (5.8). It is necessarily independent of the stress $\bar{\sigma}_1$ before unloading in this model, though the initial strain e_1^+ depends on previous loading history. If appreciable dependence of the strain-rate on previous stress is observed in experimental programmes, then this differential model with response coefficients depending on current stress and strain only is not satisfactory. Suppose that the unloading responses determine a consistent $f(e)$. Let

$e \to e_\infty$ as $t \to \infty$ where

$$0 \le e_\infty < e_1^+ , \tag{5.27}$$

so that the final strain is less than that at time t_1^+, since $f(e) < 0$, and is still a compression; that is, assume that a compressive stress history followed by stress removal does not produce a stretch. Now

$$t - t_1 = \int_e^{e_1^+} \frac{de'}{f(e')} , \qquad \int_{e_\infty}^{e_1^+} \frac{de'}{f(e')} = +\infty , \tag{5.28}$$

so $f^{-1}(e)$ is integrable at e_1^+ to obtain finite t for a strain e such that $e_\infty < e < e_1^+$. The non-relaxing case $e \equiv e_1^+$, $f \equiv 0$, is excluded. Further, the unbounded integral necessary for the finite limit e_∞ requires $f^{-1}(e)$ to be non-integrable at e, and since (5.28) applies for all $e_1^+ > 0$, and hence all $e_\infty > 0$, $f^{-1}(e)$ is non-integrable at $e = 0$, but integrable at all $e_1^+ > 0$. Thus $e_\infty = 0$, so this model requires complete relaxation given sufficient time, if non-relaxation is excluded, which is also a feature of linear viscoelastic solids; partial relaxation is not possible. Hence

$$e_\infty = 0 , \quad f(e) > 0 \text{ for } e > 0 , \quad f(e) \sim f_o e^n \ (n \ge 1) \text{ as } e \to 0. \tag{5.29}$$

The case $n = 1$ is given by $\hat{\omega}$, $\hat{\phi}$, or $\hat{\omega}^*$, $\hat{\phi}^*$ bounded and non-zero as $e \to 0$. A simple example is

$$f(e) = f_o e , \quad e = e_1^+ \exp[-f_o(t-t_1)] \quad \text{for} \quad t > t_1 , \tag{5.30}$$

which couples with the constant $\hat{E}$ approximation (5.26) to give

$$\tfrac{3}{2}\hat{\phi}^{*} = E_o(1-2)^2 \ , \quad \hat{\omega}^{*} = E_o f_o (1-e)^3 \ , \quad \hat{\psi}^{*} = \frac{E_o[\,F(\bar{\sigma},e)+f_o e\,]}{\bar{\sigma}} \ , \qquad (5.31)$$

where the stress coefficient $\hat{\psi}^{*}$ in the relation (5.5) depends on the constant load response $F(\bar{\sigma},e)$. If E_o and f_o can be estimated from limited stress jump data and unloading data, then a comprehensive CL programme will determine $F(\bar{\sigma},e)$ to complete the uni-axial relation (5.5).

6. Viscoelastic tensor relations of differential type

A brief summary of the frame-indifferent differential tensor relations of fluid and solid type developed by (MS, SM) is now presented. These are the three-dimensional models from which the relation (3.1) and (4.1) in uni-axial stress are derived. The limit to which a complete uni-axial description determines the three-dimensional response is shown, and in turn the requirements of a multi-axial load test programme sufficient to determine the three-dimensional model are indicated.

In both fluid and solid models the ice is assumed to be incompressible, which is a restriction on the possible deformation, and so the mean pressure is not determined by the deformation. The differential relations therefore connect the deviatoric stress and appropriate time derivatives to strain-rate and strain-acceleration or strain and strain-rate respectively. Let $\underset{\sim}{\sigma}$ denote the Cauchy stress tensor (with diagonal components positive in tension to follow the usual convention, so a uni-axial compressive stress $\sigma = -\sigma_{11}$ for example) with rectangular components σ_{ij} $(i,j = 1,2,3)$, and $\underset{\sim}{S}$ the deviatoric stress defined by

$$\underset{\sim}{S} = \underset{\sim}{\sigma} - \frac{1}{3}(\text{tr}\ \underset{\sim}{\sigma})\underset{\sim}{1}\ , \qquad \text{tr}\ \underset{\sim}{S} = 0\ . \tag{6.1}$$

If $\underline{v}(\underline{x},t)$ is the spatial velocity field with components v_i in rectangular coordintes Ox_i, the rate of strain tensor $\underset{\sim}{D}$ is defined by

$$D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\ , \quad I_1 = \text{tr}\ \underset{\sim}{D} = \text{div}\,\underline{v} = 0\ , \tag{6.2}$$

where the latter constraint imposes incompressibility. $\underset{\sim}{\sigma}$, $\underset{\sim}{S}$, and $\underset{\sim}{D}$ are frame indifferent tensors (Truesdell 1966). The frame-indifferent strain-acceleration is given by the Rivlin-Ericksen tensor

$$\underset{\sim}{A}^{(2)} = 2\dot{\underset{\sim}{D}} + 4\underset{\sim}{D}^2 + 2(\underset{\sim}{D}\underset{\sim}{W} - \underset{\sim}{W}\underset{\sim}{D})\ , \tag{6.3}$$

where the rotation tensor $\underset{\sim}{W}$ is defined by

$$W_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)\ . \tag{6.4}$$

Similarly, a frame-indifferent deviatoric stress-rate is given by

$$\underset{\sim}{S}^{(1)} = \dot{\underset{\sim}{S}} + \underset{\sim}{S}(\underset{\sim}{D} + \underset{\sim}{W}) + (\underset{\sim}{D} - \underset{\sim}{W})\underset{\sim}{S}\ . \tag{6.5}$$

The first principal invariants of $\underset{\sim}{A}^{(2)}$ and $\underset{\sim}{S}^{(1)}$ are given by

$$\operatorname{tr} \mathbf{A}^{(2)} = 4 \operatorname{tr} \mathbf{D}^2 \ , \quad \operatorname{tr} \mathbf{S}^{(1)} = 2 \operatorname{tr}(\mathbf{S}\mathbf{D}) \ , \tag{6.6}$$

while the non-zero second and third principal invariants of $\mathbf{D}$ and $\mathbf{S}$ are

$$\begin{aligned} I_2 &= \tfrac{1}{2} \operatorname{tr} \mathbf{D}^2 \ , \qquad I_3 = \det \mathbf{D} \ , \\ J_2 &= \tfrac{1}{2} \operatorname{tr} \mathbf{S}^2 \ , \qquad J_3 = \det \mathbf{S} \ . \end{aligned} \tag{6.7}$$

The frame-indifferent fluid relation, necessarily isotropic, which reduces to the uni-axial relation (3.1) incorporates only linear dependence on the tensors $\mathbf{S}$, $\mathbf{S}^{(1)}$, and $\mathbf{A}^{(2)}$, but includes the general isotropic tensor function of $\mathbf{D}$. Thus

$$\begin{aligned} &\psi_1 \mathbf{S} + \psi_3[\mathbf{S}^{(1)} - \tfrac{2}{3} \operatorname{tr}(\mathbf{S}\mathbf{D}) \mathbf{1}] \\ &\quad = \phi_1 \mathbf{D} + \phi_2[\mathbf{D}^2 - \tfrac{2}{3} I_2 \mathbf{1}] + \phi_3[\dot{\mathbf{D}} + \mathbf{D}\mathbf{W} - \mathbf{W}\mathbf{D}] \ , \end{aligned} \tag{6.8}$$

where the $\mathbf{D}^2$ term of $\mathbf{A}^{(2)}$ is included in the ϕ_2 term, and the response coefficients ψ_1, ψ_3, ϕ_1, ϕ_2, ϕ_3, depend only on invariants of the various tensors. To obtain the necessary uni-axial forms (3.5) and (3.12), some dependence on the rate invariants $\dot{I}_2$ and $\dot{J}_2$ is required, as well as on I_2, I_3, J_2, J_3. In general, the response coefficients depend on the two independent strain-rate invariants and the two independent stress invariants, in contrast to the dependence of $\hat{\psi}_1$, $\hat{\psi}_3$, $\hat{\phi}_1$, $\hat{\phi}_2$, $\hat{\phi}_3$ on one strain-rate r and one stress σ

in the uni-axial relation (3.1). That is, uni-axial response cannot separate dependence on the two strain-rate invariants and the two stress invariants. Furthermore, the $\hat{\phi}_1$ and $\hat{\phi}_2$ terms of (3.1) appear only as a composite term, whereas the ϕ_1 and ϕ_2 terms of (6.8) represent the distinct dependence on $\underset{\sim}{D}$ and on $\underset{\sim}{D}^2$, which are non-proportional tensors in general. By construction, the trace of each tensor term in (6.8) is zero, and each tensor is symmetric, so expressed in principal axes in the absence of rotation there are only two independent relations. The third relation is the incompressibility constraint eliminated by using deviatoric (traceless) tensors in (6.8). The uni-axial reduction (3.1) is a special case of the tri-axial stress relations constructed later.

The frame-indifferent solid relation which reduces to the uni-axial relation (4.1) incorporates only linear dependence on the tensors $\underset{\sim}{S}$, $\underset{\sim}{S}^{(1)}$, but includes a general isotropic tensor function of $\underset{\sim}{D}$ and a general isotropic tensor function of the Cauchy-Green strain tensor $\underset{\sim}{B}$; that is, the solid is isotropic in the reference configuration. If X_i and x_i denote reference and current coordinates of a particle, then the deformation gradient tensor $\underset{\sim}{F}$ is given by

$$F_{ij} = \frac{\partial x_i}{\partial X_j} , \tag{6.9}$$

and

$$\underset{\sim}{B} = \underset{\sim}{F}\underset{\sim}{F}^T . \tag{6.10}$$

The principal invariants of $\underset{\sim}{B}$ are

$$K_1 = \mathrm{tr}\,\underset{\sim}{B}\,, \quad K_2 = \tfrac{1}{2}\{(\mathrm{tr}\,\underset{\sim}{B})^2 - \mathrm{tr}\,\underset{\sim}{B}^2\}, \quad K_3 = J^2 = \det \underset{\sim}{B} = 1, \tag{6.11}$$

where the latter is the incompressibility constraint, so only K_1 and K_2 are variables. The tensor relation is

$$\underset{\sim}{S} + \psi[\underset{\sim}{S}^{(1)} - \tfrac{2}{3}\,\mathrm{tr}(\underset{\sim}{S}\underset{\sim}{D})\underset{\sim}{1}] = \phi_1\underset{\sim}{D} + \phi_2[\underset{\sim}{D}^2 - \tfrac{2}{3}\,I_2\underset{\sim}{1}]$$
$$+ \omega_1[\underset{\sim}{B} - \tfrac{1}{3}\,K_1\underset{\sim}{1}] + \omega_2[\underset{\sim}{B}^2 - \tfrac{1}{3}(K_1^2 - 2K_2)\underset{\sim}{1}]\,, \tag{6.12}$$

where the response coefficients ψ, ϕ_1, ϕ_2, ω_1, and ω_2, are assumed to depend only on the stress and strain invariants J_2, J_3, K_1, and K_2. The form leading to (5.5) is obtained by dividing throughout by ψ, which supposes that the $\underset{\sim}{S}^{(1)}$ term is present. Note that any configuration which is a distortion of the isotropic reference configuration is anisotropic (SM). Again the trace of each tensor term is zero and only two independent relations are obtained. In the uni-axial reduction (4.1), a special case of the later tri-axial relations, the $\hat{\phi}$ and $\hat{\omega}$ terms are composites of the ϕ_1, ϕ_2 terms, and ω_1, ω_2 terms respectively, so (4.1) cannot separate the $\underset{\sim}{D}$ and $\underset{\sim}{D}^2$ dependence, and $\underset{\sim}{B}$ and $\underset{\sim}{B}^2$ dependence. Dependence on four invariants is also reduced to dependence on the single stress σ and single strain e, so dependences on two stress invariants and on two strain invariants are not separated. To derive the tensor jump relation analogous to (4.2) as a continuous limit of the differential relation (6.12) with ϕ_2 independent of invariant rates, it was shown (SM) that

$$\phi_2 = 0 . \tag{6.13}$$

Then $\hat{\phi}$ is the reduction of ϕ_1, and only the $\hat{\omega}$ term is a composite of the ω_1 and ω_2 terms.

Since both (6.8) and (6.12) involve two independent scalar relations, the test response required to describe the dependence of the response coefficients on two stress and two strain (rate) invariants, and to separate terms in the non-proportional $\underset{\sim}{D}$ and $\underset{\sim}{D}^2$ and $\underset{\sim}{B}$ and $\underset{\sim}{B}^2$, must incorporate two indpendent deviatoric stress components and corresponding strain (rate) components. A conventional tri-axial stress test does not provide two independent deviatoric relations (M), but a true bi-axial stress test does (M2, M5). Both will be obtained from the tri-axial stress analysis, and also the invariants domains covered by compressive stress tests. Shear tests yield direct properties of the deviatoric relations, but appear to be less practical.

7. Multi-axial loading geometries

Both the fluid relation (6.8), necessarily isotropic, and the isotropic solid relation (6.12), have common principle axes for all the terms in the absence of rotation, and can be represented by three scalar principal components. Only two of the component equations are independent since both equations have zero trace; that is, the sum of the three principal components is zero. In principle axes

$$\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} , \quad \underset{\sim}{S} = \begin{bmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & S_3 \end{bmatrix}, \tag{7.1}$$

$$S_1 = \tfrac{2}{3}\sigma_1 - \tfrac{1}{3}(\sigma_2+\sigma_3), \quad S_2 = \tfrac{2}{3}\sigma_2 - \tfrac{1}{3}(\sigma_1+\sigma_2), \quad S_3 = -(S_1+S_2), \tag{7.2}$$

$$\begin{gathered} p = -\tfrac{1}{3}(\sigma_1+\sigma_2+\sigma_3), \quad J_2 = \tfrac{1}{3}[\sigma_1^2+\sigma_2^2+\sigma_3^2 - (\sigma_2\sigma_3+\sigma_3\sigma_1+\sigma_1\sigma_2)], \\ J_3 = \tfrac{1}{27}[(2\sigma_1-\sigma_2-\sigma_3)(2\sigma_2-\sigma_1-\sigma_3)(2\sigma_3-\sigma_1-\sigma_2)]. \end{gathered} \tag{7.3}$$

This loading geometry is described as tri-axial stress, abbreviated to TS, when $\sigma_1, \sigma_2, \sigma_3$ are independent. Only two principal stress deviatoric components are independent however, and for practical purposes we would like to obtain two independent deviatoric components with only two principal stresses independent, both for control and measurement. p is the mean pressure, and J_2, J_3 are independent shear stress measures.

Supposing the test is performed with zero rotation,

$$\underset{\sim}{W} = \underset{\sim}{0}, \quad \underset{\sim}{D} = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & -(d_1+d_2) \end{bmatrix} , \quad \underset{\sim}{B} = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_1^{-2}\lambda_2^{-2} \end{bmatrix} , \tag{7.4}$$

where d_1, d_2 are principal strain-rates, and λ_1, λ_2, are principal stretches. The corresponding invariants are

$$
\begin{aligned}
I_2 &= d_1^2 + d_2^2 + d_1 d_2 \,, \quad I_3 = - d_1 d_2 (d_1 + d_2) \,, \\
K_1 &= \lambda_1^2 + \lambda_2^2 + \lambda_1^{-2} \lambda_2^{-2} \,, \quad K_2 = \lambda_1^2 \lambda_2^2 + \lambda_1^{-2} + \lambda_2^{-2} \,.
\end{aligned}
\tag{7.5}
$$

The fluid relation (6.8) yields the two independent differential relations

$$
\begin{aligned}
&\psi_1 S_1 + \psi_3 [\dot{S}_1 + \tfrac{2}{3}(S_1 d_1 - S_2 d_1 - S_1 d_2 - 2 S_2 d_2)] \\
&\qquad = \phi_1 d_1 + \tfrac{1}{3}\phi_2 [d_1^2 - 2d_2^2 - 2 d_1 d_2] + \phi_3 \dot{d}_1 \,, \\
&\psi_1 S_2 + \psi_3 [\dot{S}_2 + \tfrac{2}{3}(S_2 d_2 - S_2 d_1 - S_1 d_2 - 2 S_1 d_1)] \\
&\qquad = \phi_1 d_2 + \tfrac{1}{3}\phi_2 [d_2^2 - 2d_1^2 - 2 d_1 d_2] + \phi_3 \dot{d}_2 \,,
\end{aligned}
\tag{7.6}
$$

where the response coefficients depend on I_2, I_3, J_2, J_3, and their time derivatives in general. The solid relation (6.12) with restriction (6.13) yields

$- S_1 d_2$

$$
\begin{aligned}
&S_1 + \psi [\dot{S}_1 + \tfrac{2}{3}(S_1 d_1 - S_2 d_1 {}_{\wedge} - 2 S_2 d_2)] \\
&\qquad = \phi_1 d_1 + \tfrac{1}{3}\omega_1 [2\lambda_1^2 - \lambda_2^2 - \lambda_1^{-2}\lambda_2^{-2}] + \tfrac{1}{3}\omega_2 [2\lambda_1^4 - \lambda_2^4 - \lambda_1^{-4}\lambda_2^{-4}] \,, \\
&S_2 + \psi [\dot{S}_2 + \tfrac{2}{3}(S_2 d_2 - S_2 d_1 - S_1 d_2 - 2 S_1 d_1)] \\
&\qquad = \phi_1 d_2 + \tfrac{1}{3}\omega_1 [2\lambda_2^2 - \lambda_1^2 - \lambda_1^{-2}\lambda_2^{-2}] + \tfrac{1}{3}\omega_2 [2\lambda_2^4 - \lambda_1^4 - \lambda_1^{-4}\lambda_2^{-4}] \,,
\end{aligned}
\tag{7.7}
$$

where the response coefficients depend on K_1, K_2, J_2, and J_3.

Conventional tri-axial stress refers to the case $\sigma_3 = \sigma_2$; that is, to a transversely isotropic stress geometry (in the plane transverse to the axial stress σ_1), which has two independent stresses σ_1 and σ_2, and so designated bi-axial stress in (M). Here the precise description, abbreviated to TIS, is introduced. Now

$$
\begin{aligned}
&S_1 = \tfrac{2}{3}(\sigma_1 - \sigma_2), \quad S_2 = S_3 = -\tfrac{1}{2}S_1 , \\
&p = -\tfrac{1}{3}(\sigma_1 + 2\sigma_2), \; J_2 = \tfrac{1}{3}(\sigma_1 - \sigma_2)^2, \; J_3 = \tfrac{2}{27}(\sigma_1 - \sigma_2)^3 = \pm 2(J_2/3)^{\frac{3}{2}} ,
\end{aligned}
\tag{7.8}
$$

so there is only one independent deviatoric stress and one independent deviatoric stress invariant. Similarly, there is only one independent strain-rate d_1 and one independent stretch λ_1, with

$$
\begin{aligned}
&d_3 = -(d_1 + d_2) = d_2 \Rightarrow d_3 = d_2 = -\tfrac{1}{2}d_1 . \\
&\lambda_3 = \lambda_1^{-1}\lambda_2^{-1} = \lambda_2 \Rightarrow \lambda_3 = \lambda_2 = \lambda_1^{-\frac{1}{2}} , \\
&I_{\cancel{3}2} = \tfrac{3}{4}d_1^2 , \quad I_3 = \tfrac{1}{4}d_1^3 = \pm 2(I_2/3)^{\frac{3}{2}} , \\
&K_1 = \lambda_1^2 + 2\lambda_1^{-1} , \quad K_2 = 2\lambda_1 + \lambda_1^{-2} ,
\end{aligned}
\tag{7.9}
$$

so K_1 determines λ_1 and hence determines K_2. With the identities each term of $(7.6)_2$ is a multiple $(-\frac{1}{2})$ of the corresponding term of $(7.6)_1$, and similarly for (7.7), so only one independent relation is obtained.

Uni-axial stress, abbreviated to US, is given by setting

$\sigma_3 = \sigma_2 = 0$ in TIS, so

$$S_1 = \tfrac{2}{3}\sigma_1 , \quad S_3 = S_2 = -\tfrac{1}{2}S_1 , \quad J_2 = \tfrac{1}{3}\sigma_1^2 , \quad J_3 = \pm\ 2(J_2/3)^{\frac{3}{2}} , \qquad (7.10)$$

and the strain-rate and strain expressions are unchanged. The one independent relation of (7.6) and that of (7.7) have common forms for TIS and US, namely

$$\psi_1 S_1 + \psi_3[\dot{S}_1 + S_1 d_1] = \phi_1 d_1 + \tfrac{1}{2}\phi_2 d_1^2 + \phi_3 \dot{d}_1 , \qquad (7.11)$$

$$S_1 + \psi[\dot{S}_1 + S_1 d_1] = \phi_1 d_1 + \tfrac{2}{3}\omega_1(\lambda_1^2 - \lambda_1^{-1}) + \tfrac{2}{3}\omega_2(\lambda_1^4 - \lambda_1^{-2}) , \qquad (7.12)$$

where S_1 is given by $(7.8)_1$ and $(7.10)_1$ respectively. The US relations (3.1) and (4.1) are recovered by using the transformations

$$\sigma = -\sigma_1 , \ r = -\ d_1 , \quad \hat{\omega} e = \omega_1[1-(1-e)^3](1-e) + \omega_2[1-(1-e)^6]. \ (7.13)$$

together with the relations $(2.2)_3$ and (2.3). The paths in the invariants domains, $J_3 = \pm\ 2(J_2/3)^{3/2}$ and $I_3 = \pm\ 2(I_2/3)^{3/2}$, are also common to TIS and US, so no extra freedom is obtained from the two independent stresses σ_1, σ_2 of TIS. In fact, with both σ_1 and σ_2 negative (compressive), the ranges of J_2, J_3 for given σ_1 are maximum when $\sigma_2 = 0$; that is, for US.

Finally, bi-axial stress, abbreviated to BS, will refer to the case of independent axial stress σ_1 and one lateral stress σ_2, with the second lateral direction stress free,

$\sigma_3 = 0$. Now

$$S_1 = \frac{2}{3}\sigma_1 - \frac{1}{3}\sigma_2 \quad , \quad S_2 = \frac{2}{3}\sigma_2 - \sigma_1 \quad ,$$

$$p = -\frac{1}{3}(\sigma_1+\sigma_2), \; J_2 = \frac{1}{3}(\sigma_1^2+\sigma_2^2-\sigma_1\sigma_2), \; J_3 = \frac{1}{27}(\sigma_1+\sigma_2)(\sigma_1-2\sigma_2)(2\sigma_1-\sigma_2), \tag{7.14}$$

so that S_1 and S_2 are independent, and J_2 and J_3 are independent. Also the two strain-rates d_1 and d_2 are independent, and the two stretches λ_1 and λ_2 are independent. Hence (7.6) and (7.7) each yield two independent differential relations between $\sigma_1,\sigma_2,\dot{\sigma}_1,\dot{\sigma}_2,d_1,d_2$, and $\dot{d}_1,\dot{d}_2$, or λ_1,λ_2, respectively. In (7.6) the ϕ_1 and ϕ_2 terms are independent, and in (7.7) the ω_1 and ω_2 terms are independent, not lost in BS, so dependence on $\underset{\sim}{D}$ and on $\underset{\sim}{D}^2$ is distinguished, and dependence on $\underset{\sim}{B}$ and on $\underset{\sim}{B}^2$ is distinguished. Thus BS provides two geometrically independent components of the tensor relation, and with incompressibility completes the three-dimensional description of an isotropic material. If anisotropic solid models are required, then BS with respect to sufficient different reference axes will be necessary to provide a full description.

The analysis of constant stress and constant strain-rate responses in BS for the fluid relations (7.6) may, or may not, be a generalisation of the uni-axial analysis detailed in section 3. There the non-monotonicity of strain-rate and stress respectively gave rise to two-branch solutions of the differential equations with implications for the response coefficients. We have no knowledge of the response shapes in BS at present, so cannot proceed along the same lines. In principle, this truly two dimensional response must distinguish the $\underset{\sim}{D}$ and $\underset{\sim}{D}^2$ terms, but it cannot be stated whether the response overdetermines the model as in US, or underdetermines the model. If the latter, then further types of test in BS, independent of constant stress and constant strain-rate responses, would be required, or, alternatively, a reduced fluid model constructed. However, the US response suggests that the model is not underdetermined by these tests.

A BS analysis of the solid relations (7.7) has not yet been performed, but may be a more direct extension of the US analysis since two-branch solutions are not involved. However, replacement of time by strain in the functional descriptions is not unique, since there are now two independent strain components, but axial strain would be the first choice.

It is possible that constant lateral strain rate (d_2 or $\dot{\lambda}_2$) may not be practical, and a mixed loading of constant d_1 with $d_2 = 0$, or constant $\dot{\lambda}_1$ with $\lambda_2 = 1$, representing a lateral constraint, may be prefered. The associated analyses of the BS relations will be more complex.

8. Dependence domains in stress invariants space

The response coefficients in the tensor relations (6.8) and (6.12) depend on the shear stress invariants J_2 and J_3, and in (6.8) on their rates and on strain-rate invariants I_2 and I_3 and their rates, and in (6.12) on the strain invariants K_1 and K_2. In practice, tests will be restricted largely (or entirely) to compressive stresses, $\sigma_1 \leq 0$, $\sigma_2 \leq 0$, $\sigma_3 \leq 0$. This limits, in general, the domain of the (J_2,J_3) plane covered by each of the configurations US, TIS, and BS, and in turn the domain in the (I_2,I_3) plane or (K_1,K_2) plane, so that response coefficients may not be determined over a range of their arguments sufficient to cover all practical load histories. The corresponding situation for dependence on mean pressure and on shear stress invariant is also discussed and compared.

Consider the compressive stress restriction

$$\sigma_1(t) \leq 0 \ , \quad \sigma_2(t) = \eta(t)\sigma_1(t) \leq 0 \ , \quad \eta(t) \geq 0 \ , \tag{8.1}$$

together with

$$\text{US: } \eta = 0 \ , \ \sigma_3 = 0, \quad \text{TIS: } \ \sigma_3 = \sigma_2 \ , \quad \text{BS: } \ \sigma_3 = 0 \ . \tag{8.2}$$

It is convenient to express the invariants as functions of σ_1 and η with each constant η denoting a ray in the (σ_1,σ_2) plane (fourth quadrant). In US only a single ray $\sigma_2 = 0$ $(\eta = 0)$ is covered, and J_2, J_3 are given by (7.10) with $\sigma_1 < 0$,

$$\text{US: } \ J_2 = \tfrac{1}{3}\sigma_1^2 \geq 0 \ , \quad J_3 = -\,2(J_2/3)^{3/2} \leq 0 \ , \tag{8.3}$$

so only one branch of the curve in the (J_2,J_3) plane given by $(7.10)_4$ is realised, shown in Fig. 6. From (7.8), for TIS,

$$\text{TIS:}\quad J_2 = \tfrac{1}{3}\sigma_1^2(1-\eta)^2 \geq 0, \quad J_3 = \tfrac{2}{27}\sigma_1^3(1-\eta)^3 = \pm 2(J_2/3)^{\frac{3}{2}} \text{ as } \eta \lessgtr 1, \quad (8.4$$

so both branches of the curve are realised by $\sigma_2 > \sigma_1$ and $\sigma_2 < \sigma_1$ respectively, and the upper branch defines the uni-axial tension configuration. It is, though, only one curve in the (J_2,J_3) plane, Fig. 6. In contrast, for BS (7.14) yields

$$\text{BS:}\quad J_2 = \tfrac{1}{3}\sigma_1^2(\eta^2-\eta+1) \geq 0, \quad J_3 = \frac{\sigma_1^3}{27}(2\eta^3-3\eta^2-3\eta+2), \tag{8.5}$$

with independent $\sigma_1 \leq 0$ and $\eta \geq 0$. Define

$$\bar{J}_2 = \frac{3J_2}{\sigma_1^2} = \frac{3}{4} + (\eta-\tfrac{1}{2})^2 ,$$

$$\bar{J}_3 = \frac{27J_3}{2\sigma_1^3} = -\ (\eta-\tfrac{1}{2})[\tfrac{9}{4} - (\eta-\tfrac{1}{2})^2], \tag{8.6}$$

$$k(\eta) = \frac{\bar{J}_3}{\bar{J}_2} = \frac{9J_3}{2J_2\sigma_1} , \quad k(0) = 1 ,$$

then $(8.6)_{2,3}$ imply a relation $\bar{J}_3 = h(\bar{J}_2)$ obtained by eliminating η , not necessarily single-valued. This corresponds to

$$J_3 = \frac{2\sigma_1^3}{27}\, h\left(\frac{3J_2}{\sigma_1^2}\right) , \tag{8.7}$$

so that J_3 depends on both J_2 and σ_1 in general. A single curve in the (J_2, J_3) plane is obtained if, and only if, $(\partial J_3/\partial\sigma_1)|J_2 = 0$; that is, at fixed J_2 there is one value J_3 though both J_2 and J_3 vary as σ_1 varies. Then h satisfies the diferential equation

$$\bar{J}_2 h'(\bar{J}_2) = \frac{3}{2} h(\bar{J}_2) \ , \tag{8.8}$$

with solution

$$\bar{J}_3 = K \bar{J}_2^{\frac{3}{2}} \ , \tag{8.9}$$

trivially satisfied by the TIS relations (8.4), but clearly not a property of the BS relations $(8.6)_{1,2}$, for which, in particular, at $\eta = \frac{1}{2}$, $\bar{J}_2 = \frac{3}{4}$, $\bar{J}_3 = 0$. Hence (8.5) determines a finite (J_2, J_3) domain for a given range of $\sigma_1 \leq 0$, $\eta \geq 0$.

Both $\bar{J}_2(\eta)$ and $\bar{J}_3(\eta)$ are symmetric about $\eta = \frac{1}{2}$, and so also $k(\eta)$. $\bar{J}_2(\eta)$ has a single minimum $\frac{3}{4}$ at $\eta = \frac{1}{2}$ and is strictly positive. $\bar{J}_3(\eta)$ has zeros at η_i $(i = 1,2,3)$, and a maximum at η_M and minimum at η_m, given by

$$(\eta_1, \eta_2, \eta_3) = (-1, 0{\cdot}5, 2), \ \eta_M = \frac{1}{2}(1+3^{\frac{1}{2}}), \ \eta_m = -\frac{1}{2}(3^{\frac{1}{2}}-1), \tag{8.10}$$

with

$$\bar{J}_3(\eta_M) = \frac{3}{4}(3)^{\frac{1}{2}}, \quad \bar{J}_3(\eta_m) = -\frac{3}{4}(3)^{\frac{1}{2}} \ , \tag{8.11}$$

and $k(\eta)$ has a maximum k_M at η_ℓ and minimum k_m at η_s given by

$$\eta_\ell \approx -0{\cdot}1 \ , \quad k_M \approx 1{\cdot}02 \ , \quad \eta_s \approx 1{\cdot}1 \ , \quad k_m \approx -1{\cdot}02. \tag{8.12}$$

Also

$$\bar{J}_2(\eta) \sim \eta^2 \; , \quad \bar{J}_3(\eta) \sim \eta^3 \; , \; k(\eta) \sim \eta, \text{ as } \quad \eta \to \pm\infty . \qquad (8.13)$$

Figure 7 shows sketches of $\bar{J}_2(\eta)$, $\bar{J}_3(\eta)$, and $k(\eta)$. Specifying η determines $\bar{J}_2$, $\bar{J}_3$, k, then specifying σ_1 determines J_2 and J_3, so values of the test parameters σ_1, η determine a point in the (J_2, J_3) plane. If J_2 and J_3 are specified, then a 6-tuple equation for η (with probably more than one real root) must be solved numerically, then σ_1 determined, but this does not arise in data correlation.

It remains to determine the domain of the (J_2, J_3) plane, Fig. 6, covered by compressive BS tests $\sigma_1 \le 0$, $\eta \ge 0$. While $\bar{J}_3$ takes all values greater than $\bar{J}_3(2) \approx -\frac{1}{2}(3^{\frac{1}{2}} - 1)$ as η is varied, and $\bar{J}_2$ all values greater than $\frac{3}{4}$, with $\bar{J}_3 = k(\eta)\bar{J}_2$, J_2 and J_3 depend also on σ_1. Eliminating σ_1 between the expressions (8.5) shows that

$$J_3 = -\,2k(\eta)(J_2/3)^{3/2} \; , \qquad (8.14)$$

so that at fixed stress ratio η, hence fixed k, as σ_1 and σ_2 are increased accordingly from zero a curve in the (J_2, J_2) plane similar to one of the TIS branches is obtained, the choice depending on the sign of k. Negative J_3 corresponds to $k > 0$, and approach to the limit line $J_2 \to 0$, $J_3 < 0$ requires large k, hence large $\eta = \sigma_2/\sigma_1$, while σ_2 and σ_1 approach zero. In practice a limit curve of the form (8.14) may not apply as J_2, $J_3 \to 0$ since the large η may not be maintained, so Fig. 6 shows only an estimate of the limit curve for negative J_3. It is clearly much closer to the limit $J_2 \to 0$ than the TIS, US compression curve.

Positive J_3 corresponds to $k < 0$, and here there is a precise limit $k_m \approx -1{\cdot}02$, with limit curve

$$J_3 = -2k_m (J_2/3)^{3/2} \approx 2{\cdot}04 (J_2/3)^{3/2} , \tag{8.15}$$

which extends the domain only slightly beyond the TIS, uni-axial tension, branch, Fig. 6. Since values of k in the range $k_m < k \leq 1$ occur for two values of η, the same curve (8.14) is repeated, and consistency of the corresponding data is necessary to justify the assumed model. In applications which have approximately a compressive plane stress configuration, the domain covered by BS data is all that is needed, and if there is a principal stress in the third direction which is compressive, the corresponding (J_2, J_3) domain will not extend to the tension limit of BS.

While the incompressibility approximation used in the model relations will be good in many applications, ductility may be influenced by mean pressure. For example, the constant strain-rate response in uni-axial compressive stress may exhibit an increased peak stress σ_M if conducted under a superposed isotropic pressure. That is, deviatoric (shear) response, may be influenced by mean pressure, even though the pressure is not determined by the deformation history. To determine dependence on the three stress invariants p, J_2, J_3, will require general tri-axial stress data, the response to three independent stresses. Alternatively, if response only to two independent stresses is practical, this can be interpreted by dependence of response coefficients on pressure and one shear stress invariant in place of two shear stress invariants. The additional dependence on I_2, I_3, or K_1, K_2, can be retained, since $I_1 = \text{tr}\ \underset{\sim}{D} = 0$ and $K_3 = \det \underset{\sim}{F} = 1$ by incompressibility. Consider dependence on (p, J_2), and the domains of the (p, J_2) plane covered by US, TIS, BS, for compressive stresses which imply $p \geq 0$.

Thus, with $\sigma_1 \leq 0$, $\sigma_2 = \eta\sigma_1 \leq 0$, from (7.8) and (7.14),

$$\text{US: } p = -\tfrac{1}{3}\sigma_1 \geq 0 , \quad J_2 = \tfrac{1}{3}\sigma_1^2 = 3p^2 , \tag{8.16}$$

$$\begin{aligned} &\text{TIS: } p = -\tfrac{1}{3}\sigma_1(1+2\eta) \geq 0, \quad J_2 = \tfrac{1}{3}\sigma_1^2(1-\eta)^2 = \mu(\eta)p^2 , \\ &\qquad \mu(\eta) = 3\left(\frac{1-\eta}{1+2\eta}\right)^2 \geq 0 , \end{aligned} \tag{8.17}$$

$$\begin{aligned} &\text{BS: } p = -\tfrac{1}{3}\sigma_1(1+\eta) \geq 0 , \quad J_2 = \tfrac{1}{3}\sigma_1^2(1-\eta+\eta^2) = \nu(\eta)p^2 , \\ &\qquad \nu(\eta) = \frac{3(1-\eta+\eta^2)}{(1+\eta)^2} \geq 0 . \end{aligned} \tag{8.18}$$

Again US covers only a single curve in the (p, J_2) plane, but both TIS and BS cover finite domains as the ratio η is varied. However, neither TIS nor BS can realise a uni-axial tension configuration $J_2 = 3p^2$, $p = -\tfrac{1}{3}\sigma_1 < 0$, since $p \geq 0$ for all $\eta \geq 0$, $\sigma_1 < 0$ in (8.17) and (8.18). In contrast, both TIS and

BS cover uni-axial tension in the (J_2, J_3) plane.

For TIS, from (8.17), as $(-\sigma_1, -\sigma_2)$ are increased from zero at fixed ratio η, a parabola $J_2 = \mu p^2$ is covered in the (p, J_2) plane, Fig. 8. Now

$$\frac{d\mu}{d\eta} \begin{matrix} \leq \\ \geq \end{matrix} 0 \quad \text{as} \quad \begin{matrix} 0 \leq \eta \leq 1 \\ \eta > 1 \end{matrix} \; ,$$
$$\mu(0) = 3 \, , \qquad \mu_{min} = \mu(1) = 0 \, , \quad \mu \rightarrow \frac{3}{4} \quad \text{as} \quad \eta \rightarrow \infty \, , \tag{8.19}$$

so the range $0 \leq \mu \leq 3$ is covered by $0 \leq \eta \leq 1$, and the range $0 < \mu < \frac{3}{4}$ is repeated by $\eta > 1$. Consistency of the corresponding data is required to justify the (p, J_2) dependence in the assumed model. Thus TIS covers a domain in the (p, J_2) plane bounded by the positive p axes and limit parabola $J_2 = 3p^2$, shown in Fig. 8. The limit parabola is the uni-axial compression configuration.

For BS, from (8.18), the parabola $J_2 = \nu p^2$ is covered at constant η . Now

$$\frac{d\nu}{d\eta} \begin{matrix} \leq \\ > \end{matrix} 0 \quad \text{as} \quad \begin{matrix} 0 \leq \eta \leq 1 \\ \eta > 1 \end{matrix} \; ,$$
$$\nu(0) = 3, \qquad \nu_{min} = \nu(1) = \frac{3}{4} \, , \quad \nu \rightarrow 3 \quad \text{as} \quad \eta \rightarrow \infty \quad , \tag{8.20}$$

so the entire range $\frac{3}{4} < \nu < 3$ is duplicated by $\eta < 1$ and

$\eta > 1$, requiring consistency of corresponding data. Thus BS covers a domain between the limit parabolas $J_2 = \frac{3}{4}p^2$ and $J_2 = 3p^2$, shown in Fig. 8, which is more restricted than that covered by TIS, in strong contrast to their coverage of domains in the (J_2, J_3) plane. Of course, TIS still yields only one independent deviatoric relation, while BS yields two, but the excluded domains in BS, $J_2 > 3p^2$, $J_2 < \frac{3}{4}p^2$, are of practical significance. It has been shown (M5) that allowing axial tension $\sigma_1 > 0$ with lateral compression $\sigma_2 < 0$, $\eta < 0$, extends the domains to $J_2 = 3p^2$ $(p < 0)$ for TIS and BS, but still excludes the important domain $0 \le J_2 < \frac{3}{4}p^2$ for BS. Complete coverage for BS requires both axial tension and lateral tension.

9. Compressibility and dilatancy

In applications where volume changes are significant, a volume change model must be introduced, and the deviatoric relations constructed for an incompressible material must be modified appropriately. One approach is to adopt the same shear description by replacing the strain-rate, strain-acceleration or strain, in the differential relations (6.8) and (6.12) by their deviatoric parts

$$\bar{\underset{\sim}{D}} = \underset{\sim}{D} - \tfrac{1}{3}I_1\underset{\sim}{1}, \quad \dot{\bar{\underset{\sim}{D}}} = \dot{\underset{\sim}{D}} - \tfrac{1}{3}\dot{I}_1\underset{\sim}{1}, \quad \bar{\underset{\sim}{B}} = J^{-2/3}\underset{\sim}{B}, \tag{9.1}$$

then add an independent volume-change law to relate I_1 or J to stress. Now

$$\rho J = \rho_o, \qquad \rho I_1 = -\dot{\rho}, \quad J I_1 = \dot{J}, \tag{9.2}$$

by mass balance, where ρ is density with reference value ρ_o, and $J = \det \underset{\sim}{F}$ measures the density decrease factor, and $I_1 = \mathrm{tr}\, \underset{\sim}{D}$ measures the rate of increase of volume per unit current volume. In (6.8) and (6.12), the corresponding invariants for $\underset{\sim}{\overline{D}}$ and $\underset{\sim}{\overline{B}}$ are

$$\overline{I}_1 = \mathrm{tr}\, \underset{\sim}{\overline{D}} = 0 , \quad \overline{I}_2 = \tfrac{1}{2}\mathrm{tr}\, \underset{\sim}{\overline{D}}^2 = I_2 - \tfrac{1}{6}I_1^2, \quad \overline{I}_3 = \det \underset{\sim}{\overline{D}} , \tag{9.3}$$

$$\overline{K}_1 = \mathrm{tr}\, \underset{\sim}{\overline{B}} = J^{-2/3}K_1, \ \overline{K}_2 = \tfrac{1}{2}\{(\mathrm{tr}\, \underset{\sim}{\overline{B}})^2 - \mathrm{tr}\, \underset{\sim}{\overline{B}}^2\} = J^{-4/3}K_2, \ \overline{K}_3 = \det \underset{\sim}{\overline{B}} = 1,$$

so that I_1 and $\dot{I}_1$ enter the deviatoric fluid relation (6.8) explicitly, and I_1 and J enter the deviatoric solid relation (6.12) explicitly.

Elastic compression relates density to the stress invariants, mean pressure and the shear stress invariants, to satisfy frame indifference. Thus

$$\rho = g(p, J_2, J_3), \quad I_1 = -\kappa_1\dot{p} + \kappa_2\dot{J}_2 + \kappa_3\dot{J}_3 , \quad J^{-1} = g/\rho_o, \tag{9.4}$$

where the compressibility κ_1 and shear-rate coefficients depend on p, J_2, J_3. If ρ depends on pressure alone, then $\kappa_2 = \kappa_3 = 0$ and $\kappa_1 = \kappa_1(p)$. An infinitesimal volume change approximation then gives constant κ_1. The density relation (9.4) is reversible, and implies density jumps occur when jumps in the stress invariants in the argument of g are applied. When (9.4) is added to

the modified (6.8), stress-acceleration terms are introduced through $\dot{I}_1$ and $\dot{\underset{\sim}{\overline{D}}}$, but are zero for constant stress and constant strain-rate relations. $\dot{I}_1$ does not occur in the modified (6.12).

A model for dilatancy, the opening of pores and cracks under maintained shear stress, is given by

$$I_1 = h(J_2, J_3^2) \geq 0 \ , \tag{9.5}$$

defining a constant rate of volume increase per unit volume at constant deviatoric stress, and a simplified form excludes dependence on J_3. Similarly, if the ice has bulk viscosity which depends on pressure

$$I_1 = -\ell(p) \leq 0 \ , \tag{9.6}$$

giving a constant compression-rate at constant pressure. (9.5) and (9.6) may be combined additively, or I_1 expressed as a function of J_2, J_3, and p. However, there must be bounds on the maximum and minimum densities, ρ_M and ρ_m, independent of load duration but possibly depending on stress level. Approximate generalisations of (9.5) and (9.6) are

$$I_1 = q(J)h(J_2, J_3), \quad I_1 = -q(J)\ell(p) \ , \tag{9.7}$$

or some combination, where

$$q(\xi) = \left\{ \begin{array}{ll} 0 \quad \text{for} \quad \xi \leq J_M \quad \text{and} & \xi \geq J_m \\ > 0 \quad \text{for} \quad J_M < \xi < J_m & \end{array} \right\} \tag{9.8}$$

controls the permitted density range and the rates of change as the limits $J_M = \rho_o/\rho_M$ and $J_m = \rho_o/\rho_m$ are approached. The relations (9.7) are differential equations for J when I_1 is eliminated by $(9.2)_3$. No density jumps occur when stress jumps are applied in the models (9.5) - (9.7).

The response coefficients of the modified (6.8) and (6.12) may also depend on I_1 (or J) and p, so that three independent stress loading data is required in general. We can though investigate how the interpretation of US, TIS, and BS response is affected by volume change. In US, $p = -\frac{1}{3}\sigma_1 = -\frac{1}{2} S_1$ so that p and S_1 cannot be distinguished, though d_1 and d_2 are independent, and hence $\bar{d}_1$ and I_1 are independent. It is therefore possible to relate I_1 to p or to J_2 if such single dependence is assumed, but not to distinguish the possible dependences. In TIS, $p = -\frac{1}{3}(\sigma_1 + 2\sigma_2)$ and $S_1 = \frac{2}{3}(\sigma_1 - \sigma_2)$ are independent, and $\bar{d}_1$ and I_1 are independent, so that a volume change relation can be separated from the single deviatoric relation if dependence on only one shear stress invariant in addition to pressure is assumed. In BS, J_2 and J_3 are independent, d_1, d_2, and I_1 are independent, but only two of J_2, J_3, and p can be independent since only two stresses σ_1 and σ_2 can be prescribed arbitrarily. Thus, for a dilatancy model (9.5) or $(9.7)_1$ with no dependence on p, and with deviatoric response coefficients independent of p, the BS responses can be used to determine the deviatoric response over the appropriate (J_2, J_3) domain, then TIS used to determine the dilatancy response. Here the BS and TIS configurations provide complementary data.

10. Viscoelastic integral relations

Constitutive relations of the differential types proposed here incorporate histories of deformation and stress explicitly only through rates of change at the current time. The current stress for a given deformation history is obtained, though, as the solution of a differential equation in time with the integration starting from some initial instant, so there is implicit dependence on history. This arises because stress-rate is included in the differential relation, whereas conventional laws of differential type express stress explicitly in terms of deformation rates. Similarly, the current deformation is obtained as the solution of a differential equation for a given stress history. In order to describe the typical responses of ice in uni-axial stress, it is necessary that the response coefficients - coefficients of the various tensors - depend on both stress and strain or strain-rate through tensor invariants for a fluid relation and isotropic solid relation. Even the most reduced solid model left one coefficient depending on both stress and strain. Constructing practical uni-axial test programmes to obtain data over the required stress-strain or strain-rate domain is difficult, and this difficulty is compounded in the multi-axial tests to determine simultaneous dependence on two stress and two strain or strain-rate invariants.

A more general description of viscoelastic history dependence is through integral relations which express current stress explicitly as integrals of the strain history, or current strain as integrals of the stress history. The integral kernels therefore depend only on strain history or stress history respectively, and current time, and functions of stress and strain combined do not arise. Given an hereditary integral of strain-history to determine the uni-axial stress, calculation of the strain-history for a given stress history is by solving a Volterra integral equation in time for which simple numerical marching algorithms exist. Integral equation solution in three dimensions should be numerically as easy, or easier, than corresponding differential equations. Similarly, an integral expression for stress can be directly entered into the equilibrium equations for boundary-value problem formulation, yielding simultaneous integro-differential equations in place of higher order partial differential equations arising with differential operator relations. Development of integral operator relations looks an attractive programme, both to describe the viscoelastic response of ice and to improve numerical solution of boundary-value problems.

There is a well established theory of linear viscoelasticity in which the integrals are convolutions with kernels (weighting of the past strain or stress) functions of time difference only. Together with the linearity (superposition), this allows direct inversion between the strain and stress history formulations, so that either formulation uniquely determines the other. Furthermore, one constant strain test determines the stress formulation; that is, the creep function of time which is the kernel; and one constant stress test determines the strain formulation; that is, the relaxation function of time which is the kernel. The creep function and relaxation function, by the inversion, determine each other, so constant strain and constant stress responses are fully related and represent the common viscoelastic property. Recall Mellor's analogous conjecture for the constant stress and constant strain-rate responses of ice, not satisfied by the fluid and solid non-linear differential relations described earlier. To describe non-linear response it is clear that families of constant stress tests, or other families of tests, are required since the superposition property of linear response is lost. Also the convolution inversion theorem for stress and strain formulations is lost, so will inversion still exist for particular types of non-linear integral flow? If it does, then Mellor's conjecture holds for the model, and in consequence the two types of response cannot be used as independent data. The immediate corollary is whether constant stress response can be used to determine the kernel of a stress formulation, and whether constant strain-rate response can be used to determine the kernel of a strain formulation.

These questions in relation to the uni-axial response of ice have been the basis of recent research (Morland and Spring) which now has some results in preparation for publication. Various approximations of a general non-linear integral law describing fading memory have been proposed in the literature. Truncated expansions in multiple integrals with kernels depending only on the current strain represent the response to strain histories which, in some sense, have departed little from the current strain, and if the weighting factor decreases rapidly into the past, then closeness is required only in the recent past. While such models may apply even for widely varying strain histories, the motivation is lost. Also, to determine the multiple kernels, the truncation must be fixed and an appropriate number of tests performed. If further terms in the expansion are needed, the correlation procedure must be repeated for the new truncation, and the previous lower order kernels are not related to their new counterparts. Alternative models with strain-history dependent kernels allow higher order multiple integral corrections to be determined by multiple strain-step tests, retaining the lower order kernels.

For practical purposes a single integral representation is desirable, and we have now shown that the first term of the latter type of expansion is in fact also more tractable than the apparently simpler finite linear viscoelastic model given by the first term of the former expansion. That is, the numerical algorithm to calculate the stress response to constant strain-rate, given the strain formulation, is simple and fast, in contrast ~~constant~~ to a lengthy calculation for the corresponding finite linear viscoelastic model. The kernels for both are determined by a family of constant stress tests. Mellor's conjecture is therefore confirmed for this model. It is also shown that the constant strain-rate response does not lead to any direct determination of the kernel in the strain formulation, nor does the constant stress response yield the kernel in a stress formulation. The next important question is how any difference between observed and predicted constant strain-rate response could be used to determine a correction multiple integral to the strain formulation. That is, if Mellor's conjecture is not satisfied by the single integral defined by constant stress response, can an extra multiple integral term which does not change the constant stress response be determined directly from constant strain-rate response? We have not, so far, devised a direct scheme, and some optimisation procedure for approximate correlation may be the most useful approach. Detailed constant stress and constant strain-rate data is necessary before single integral models and predictions can be assessed. Discussion of integral constructions by correlation with such distinct types of response is completely absent from the present literature.

11. Concluding remarks

The use of integral relations to describe the non-linear viscoelastic response of ice is attractive for the simplicity of structure and for the absence of higher order time derivatives required in differential relations. As discussed in the previous section there is still a need for basic theoretical research on correlation of integral representations with typical types of data, and subsequently on the formulation of boundary-value problems as integro-differential equations and the development of the necessary numerical algorithms for solution construction.

Focussing on low order differential relations which are known to describe qualitiatively the typical response, there is now a need for accurate, detailed, test data in uni-axial and bi-axial stress configurations to determine the actual response coefficients. Since uni-axial data will appear first, an accurate uni-axial description should be constructed. The fluid relation is overdetermined by the anticipated constant stress and constant strain-rate data, but it has been noted how the key features can be used to determine a model. Since the solid model can account for strain-jumps, allows anisotropic extension, and is attractive for small strain applications, correlation with one or more of the reduced models analysed in section 5 should have priority. The general solid relation is underdetermined by uni-axial tests. Given a successful uni-axial relation, extrapolation to tensor relations to describe three-dimensional response can be tried in a variety of ways, to yield tentative models for preliminary solutions of field applications. Solutions may distinguish the merits of the different approximations.

The most simple model would adopt the least number of tensor terms and minimum dependence on invariants compatible with uni-axial response. As an example, extrapolating the second normalised uni-axial relation (5.5) with the simplifications (5.13) and $\phi_2^* = \omega_2^* = 0$, by assuming corresponding dependence on one deviatoric stress invariant J_2 and one strain-invariant K_1, yields a tensor relation

$$\begin{aligned} &\underset{\sim}{S}^{(1)} - \tfrac{2}{3}\mathrm{tr}(\underset{\sim}{S}\underset{\sim}{D})\underset{\sim}{1} + \psi^*(J_2,K_1)\underset{\sim}{S} \\ &\quad = \phi_1^*(K_1)\underset{\sim}{D} + \omega^*(K_1)[\underset{\sim}{B} - \tfrac{1}{3}K_1\underset{\sim}{1}] \quad , \end{aligned} \tag{11.1}$$

where ψ^*, ϕ^*, and ω^* are determined by constant strain-rate and constant stress responses, including response on unloading, in uni-axial stress. Only ψ^* depends on both stress and strain invariants. Bi-axial tests, or applications, may show that the directional simplifications leading to (11.1) are not acceptable, but at least it provides a model with appropriate qualitiative response which can be used to develop analytical and numerical techniques for boundary-value problems. Note that both constant strain-rate response and constant stress response, and the response to complete unloading from the range of stress levels considered, are necessary to construct the simplified model (11.1).

12. Terminology

e	engineering strain, decrease in length per unit initial length
r	strain-rate, rate of decrease of length per unit current length
σ	compressive traction per unit current area in uni-axial stress
$\bar{\sigma}$	compressive traction per unit initial area in uni-axial stress
CL	constant load in uni-axial stress
CLR	constant load-rate in uni-axial stress
CD	constant displacement in uni-axial stress
CDR	constant displacement-rate in uni-axial stress
ϕ, ψ, ω	response coefficients in differential relations
E	generalised Young's modulus
$\hat{E}$	stress jump-strain jump ratio in uni-axial stress
$\underset{\sim}{\sigma}$, $\underset{\sim}{S}$	Cauchy stress and deviatoric stress tensors
$\underset{\sim}{F}$, $\underset{\sim}{B}$, $\underset{\sim}{D}$	deformation gradient, Cauchy-Green strain, and strain-rate tensors
p, J_2, J_3	pressure and deviatoric stress invariants
K_1, K_2, K_3	invariants of $\underset{\sim}{B}$
I_1, I_2, I_3	invariants of $\underset{\sim}{D}$
TS	tri-axial stress
TIS	transversely isotropic stress
BS	bi-axial stress
US	uni-axial stress

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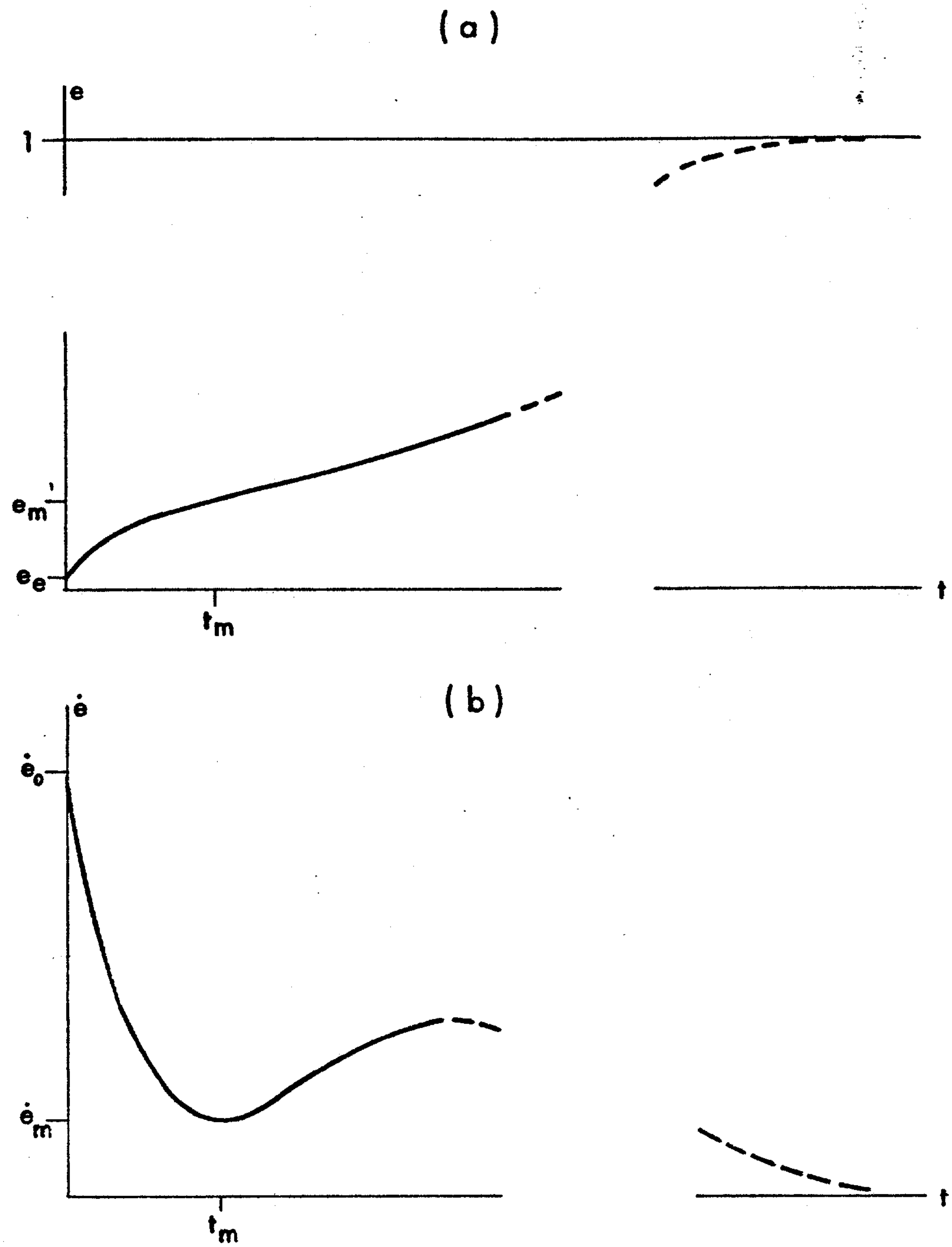

Fig. 1 - Typical response for constant load test:

(a) creep curve,
(b) strain-rate v. time.

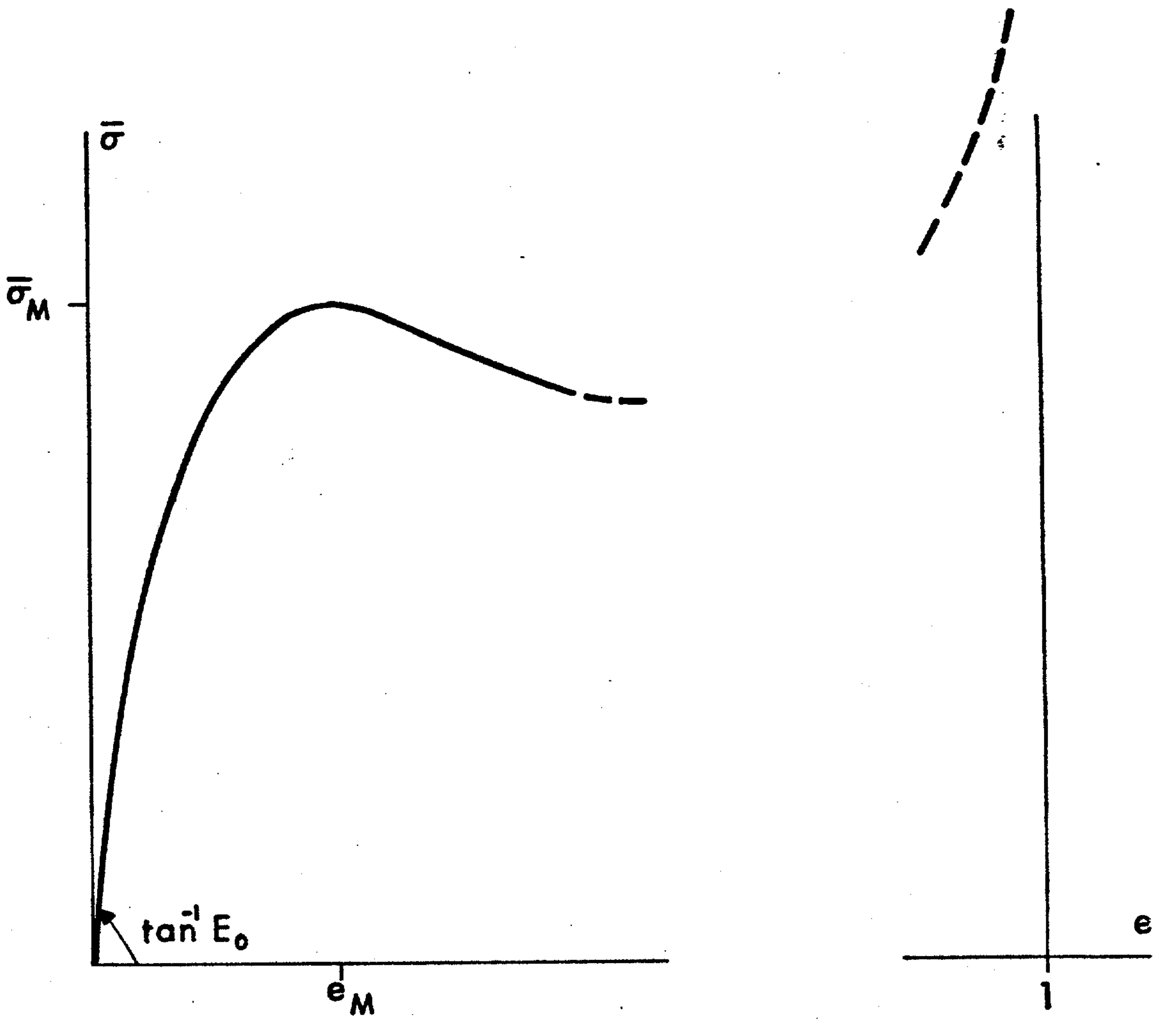

Fig. 2 - Typical stress-strain response for constant displacement rate test on ice.

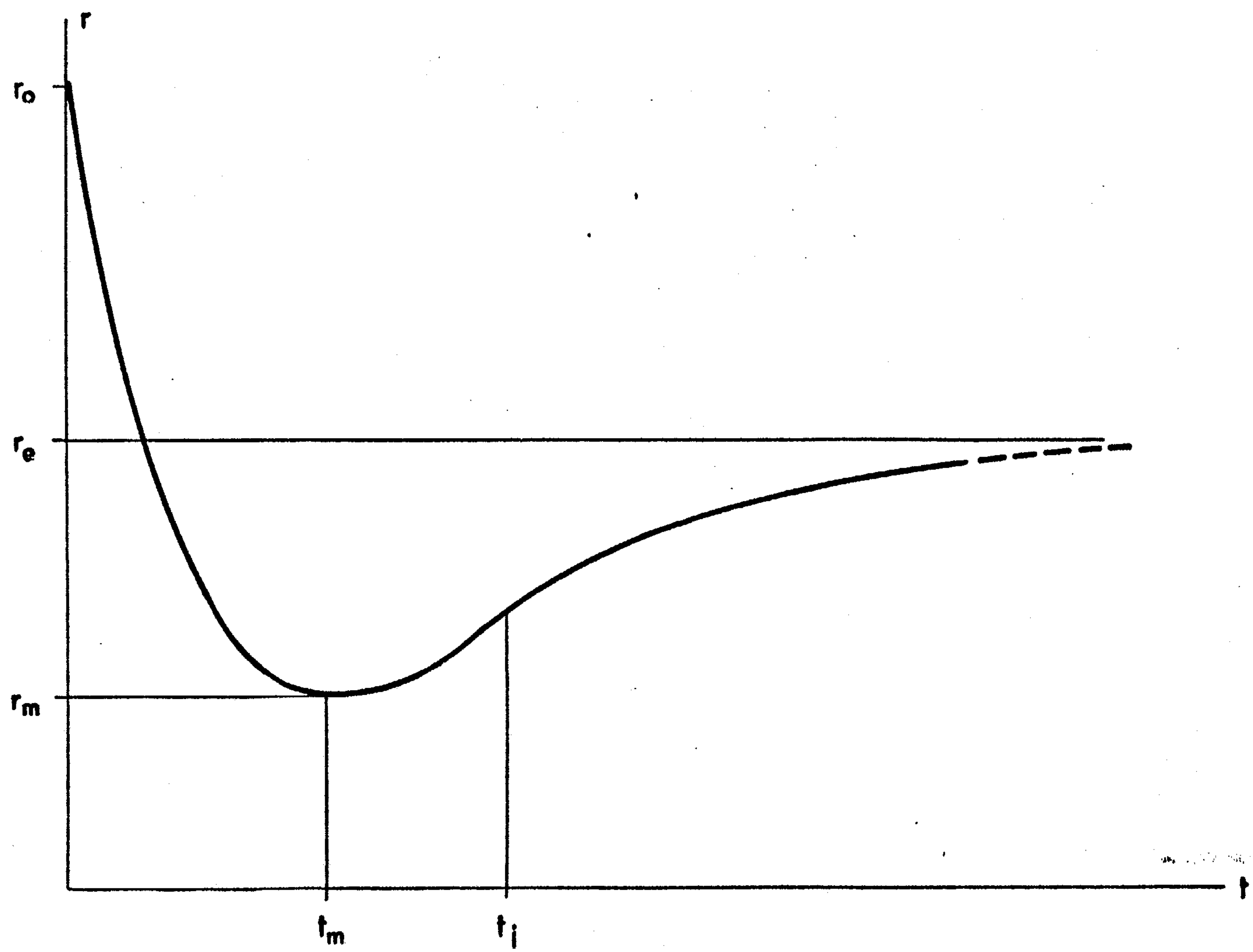

Fig. 3 - Strain-rate response at constant stress.

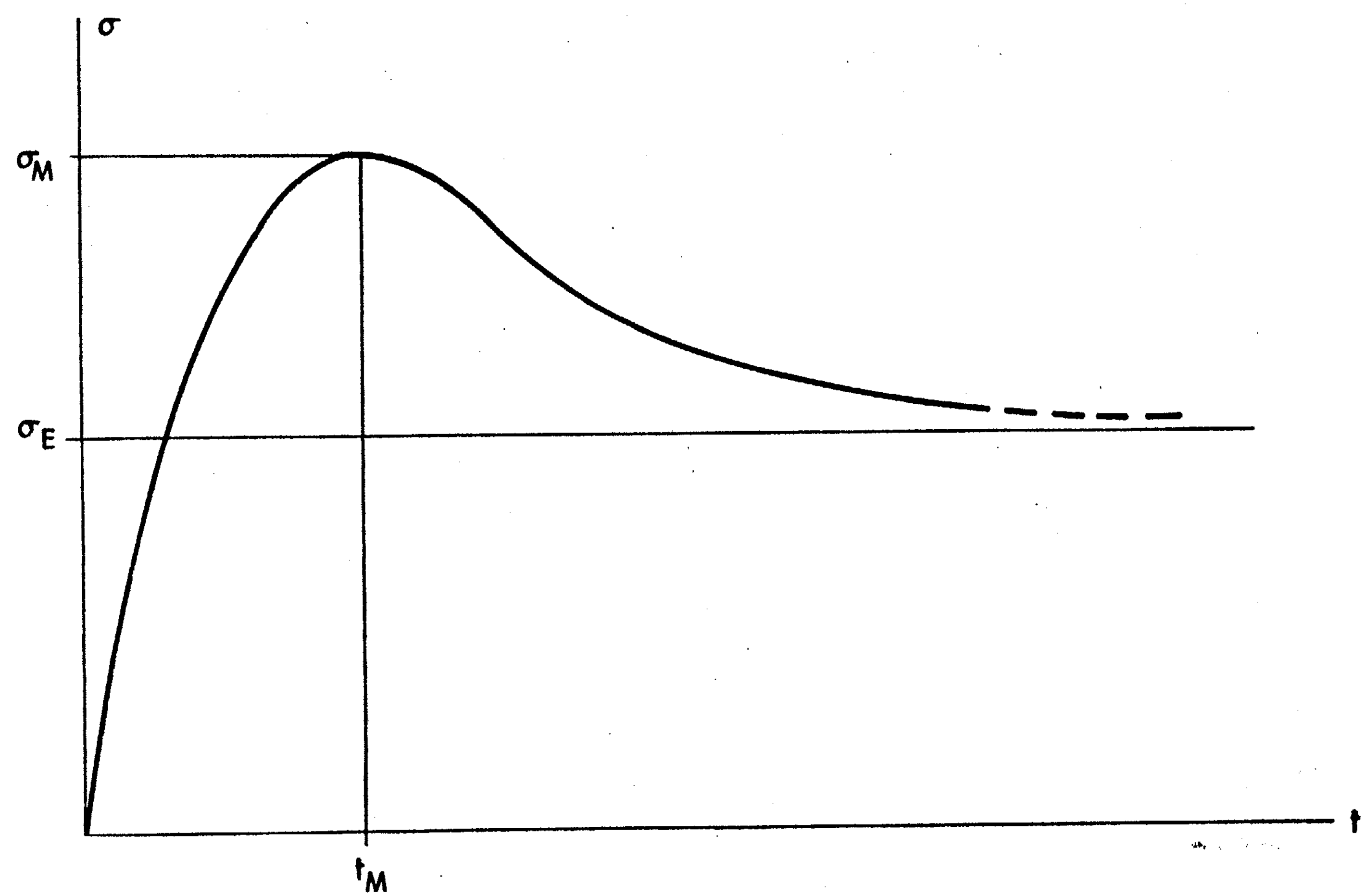

Fig. 4 - Stress response at constant strain-rate.

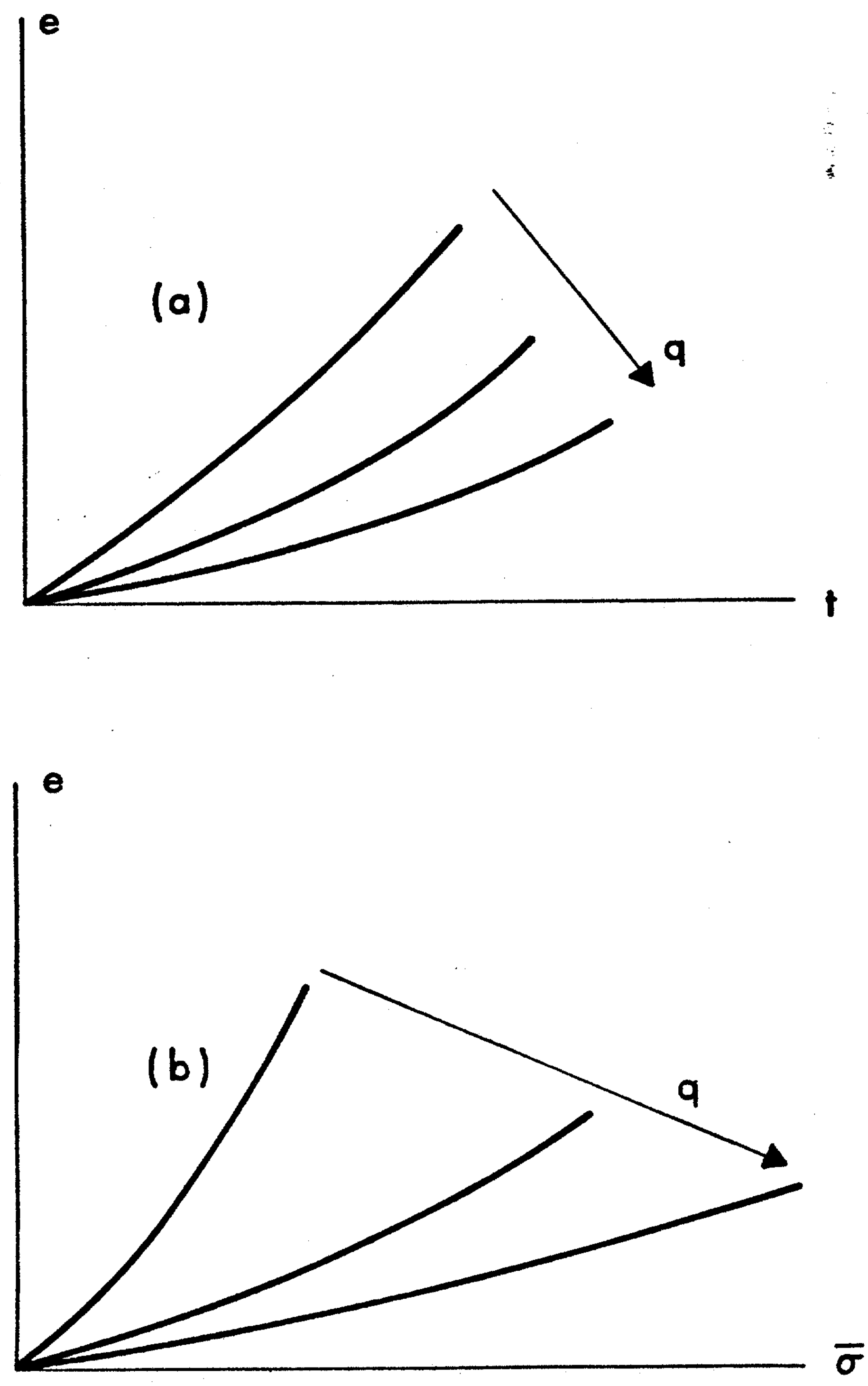

Fig. 5 - Strain response at constant load-rate,

(a) as a function of time,
(b) as a function of stress.

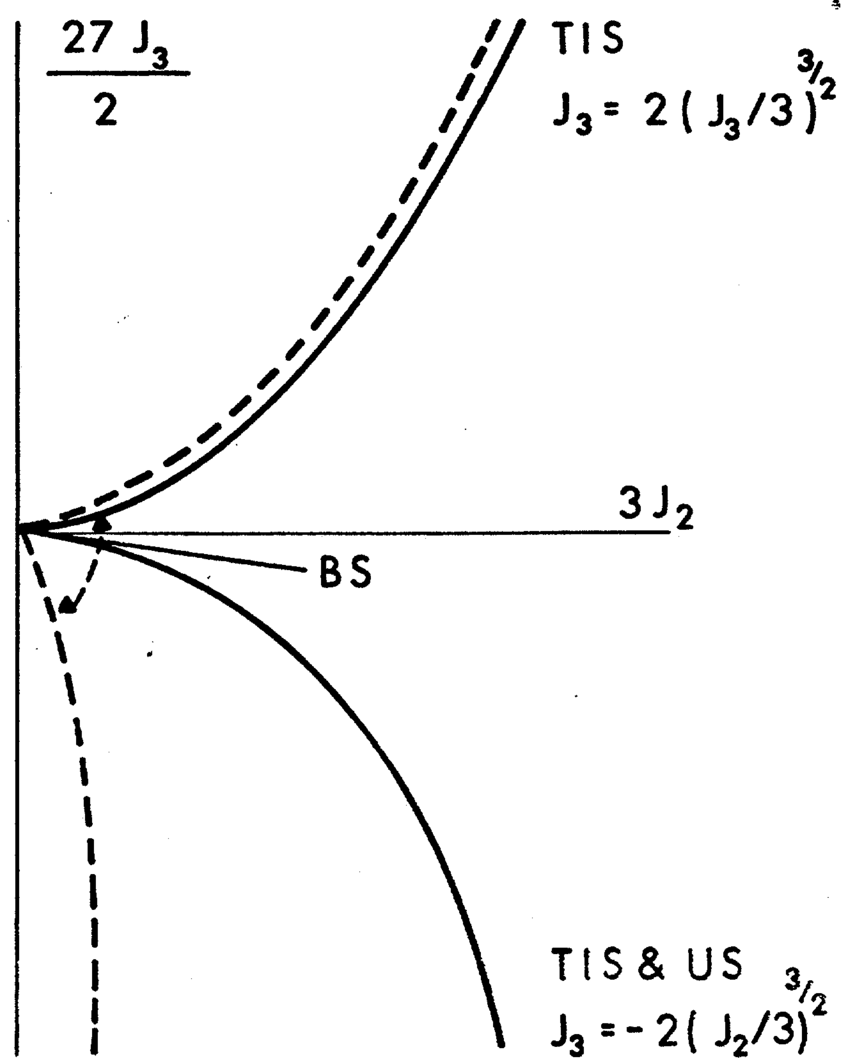

Fig. 6 - Shear invariant domains for compressive stresses.

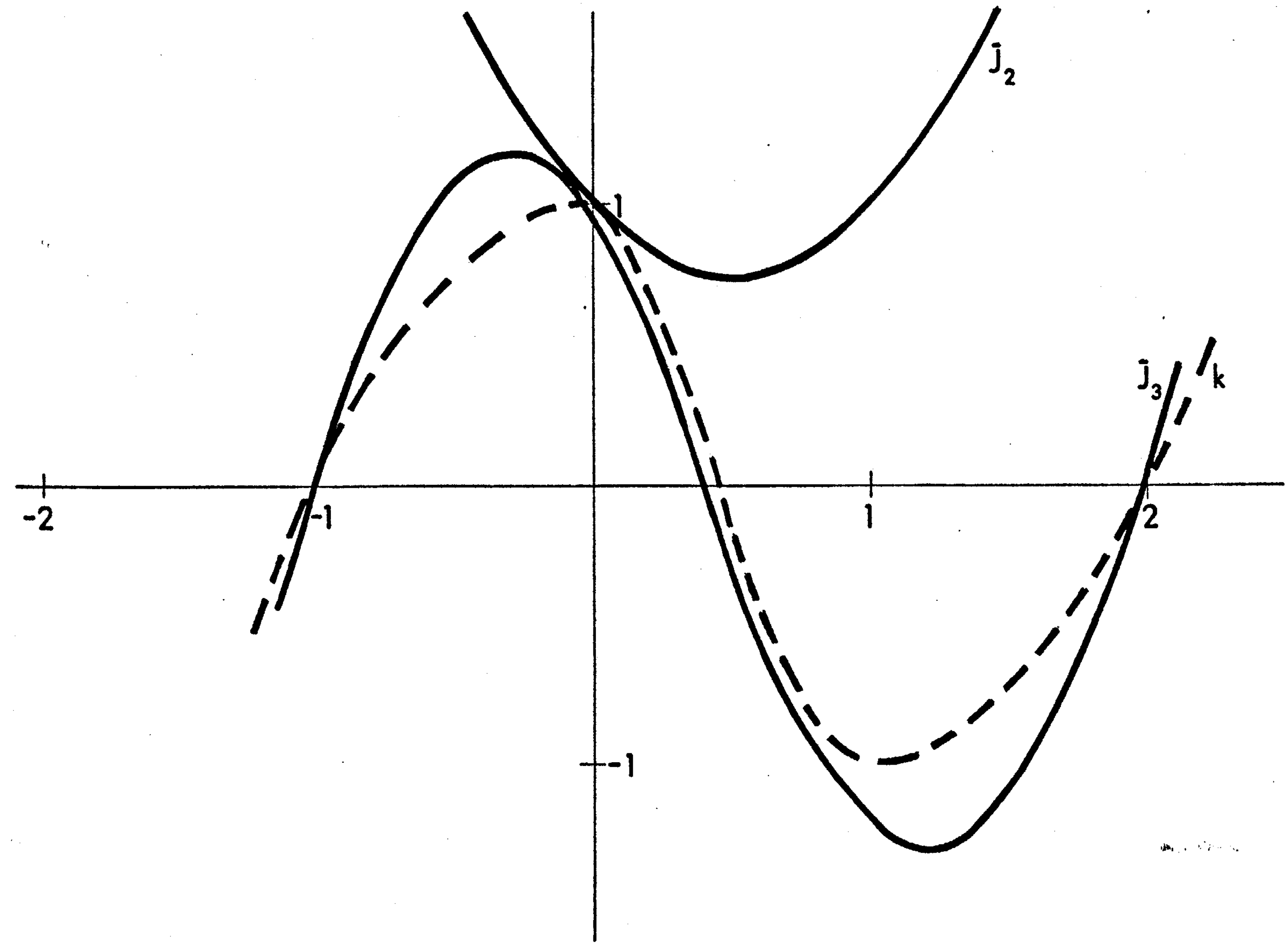

Fig. 7 - Bi-axial stress functions of the stress ratio.

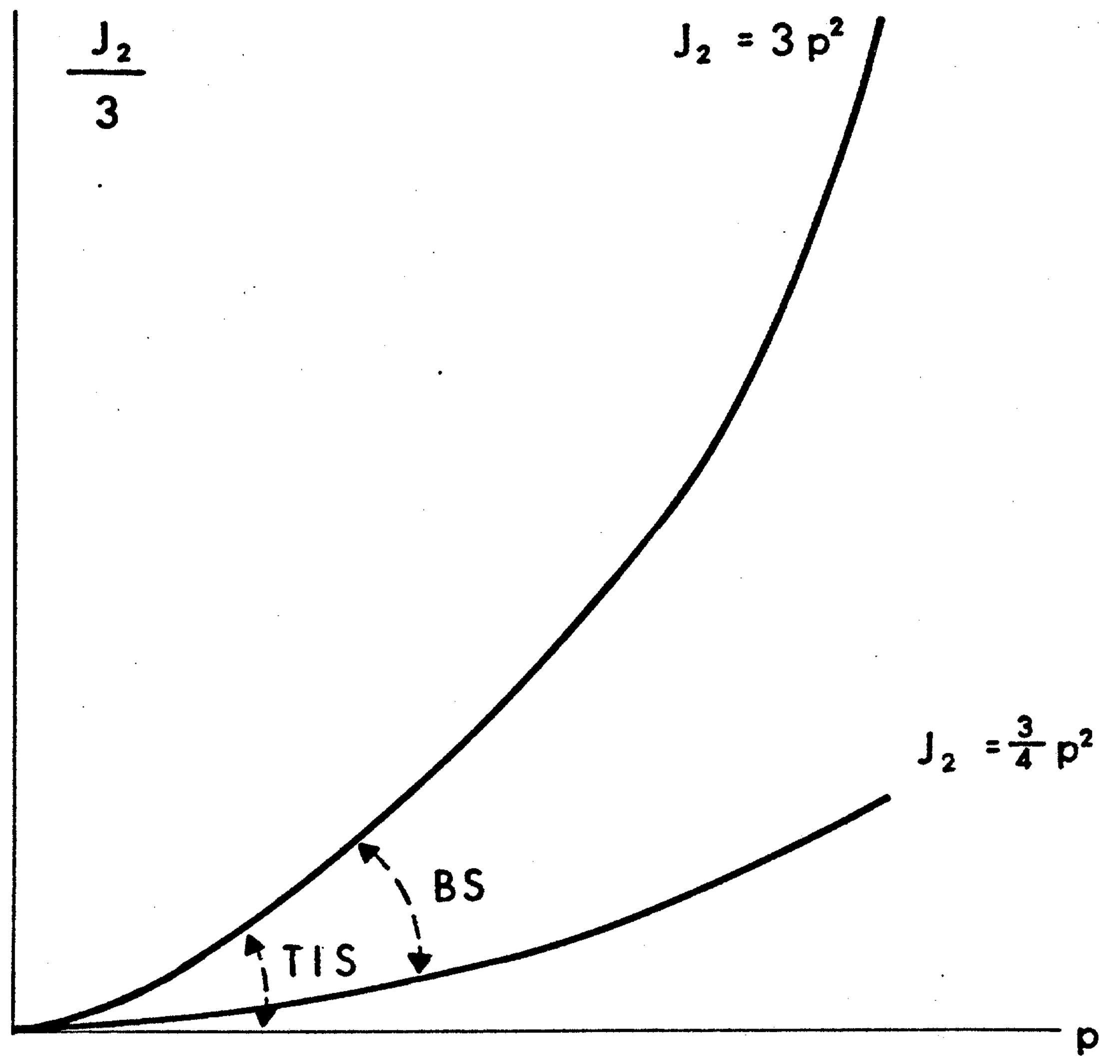

Fig. 8 - Pressure shear invariants domains for compressive stresses.

Summary of "Mechanical Properties of Sea Ice: Theoretical Phase, Sept. 1980 Nov. 1981", by L. W. Morland

A principal objective of the project Mechanical Properties of Sea Ice is to determine the constitutive (stress—strain) relations for sea ice. These relations are needed to compute ice loads on off shore structures. We believe that a better understanding of these relations will lead to a better experimental program to measure them, and will help in interpreting the measurements. For this reason L. W. Morland was hired to study potential constitutive relations from the viewpoint of applied mechanics. Reference 1 documents his efforts to date. Since it is fairly mathematical, we provide this note, which is intended to present an overview of that report in relatively nontechnical language. We will not make any attempt at completeness, nor try for mathematical rigor. Anyone interested is referred to the original report, and to published papers by Morland and Spring (2), and by Spring and Morland (3).

Morland considers both a fluid model and a solid model. Since the analysis is somewhat similar, we will only consider the solid model, which we feel to be more appropriate for engineering applications. Our approach will be slightly different from that of Ref. 1.

Uniaxial Relation

We begin, as does Morland, by considering the uniaxial case. We have the best data for this case. More importantly for our purposes, it is the easiest to understand. Thus the motivation for the equations is clearer.

We are seeking the "simplest" constitutive relation for ice. By this, we mean the mathematically simplest relation which does not violate any known physical laws, and is not inconsistant with any known behavior of ice. Since we are seeking a general relation, there will be special cases where even simpler relations will give satisfactory results. For example, in some problems linear elasticity will provide satisfactory answers, but in other cases, it is clearly inadequate.

We will cheat a little bit by making two assumptions which are not strictly correct. Our justification for these is that they greatly simplify computations, and, we hope, will not cause errors that will be significant in engineering applications. These assumptions are:

1. Ice is incompressable. This has two quite distinct implications. First, the volume does not change during deformation. Thus, if we begin with a cube of ice, and we know the change in length in two directions as a result of loading, we can compute the change in length in the other direction from the requirement of constant volume. The other implication is that none of the parameters that enter any of the equations depends in any way on pressure. This is open to some question, and below we describe how the analysis here can be easily extended to account for pressure effects, should it be necessary.

2. Ice is isotropic. While this is clearly not true for first year sheet ice, it appears to be a plausible assumption for ice from multi-year pressure ridges on a scale of interest in most engineering problems.

We begin by observing that we expect that ice under load will behave as a nonlinear viscoelastic solid. For the moment, suppose that the behavior is that of a linear viscoeleatic solid, and consider the implications in view of the known behavior of ice under load. The constitutive relation for a general linear viscoelastic solid is (see ref. 4).

$$\sigma + a_1 \dot{\sigma} + a_2 \ddot{\sigma} + \cdots = b_0 \epsilon + b_1 \dot{\epsilon} + b_2 \ddot{\epsilon} + \cdots \quad (1)$$

Here, σ is stress, ϵ is strain, and a_i and b_i are consants. The dot denotes time derivitive. By definition, a viscoelastic solid is one for which $b_0 \neq 0$ Next, recall that if a constant load is applied to ice it will creep - that is, the strain will continue to change even though the stress is constant. In this case, the only non-zero term on the left of the equation 1 is the stress, which is constant. If equation 1 is to be satisfied with a continually changing strain, then at lease one other b_i, in addition to b_0, must be non-zero. We chose b_1, and since we are seeking the simplest relation, we assume all except the first two b_is are zero. Similarly, the fact that stress will continue to decay if we apply a strain then hold it constant implies that at lease one of the a_is must be non-zero. We take a_1 to be non-zero, and all others to be zero. Since the basic argument is valid in the nonlinear case, we conclude that our relation must have stress, stress rate, strain and strain rate.

There are some other constraints on the constitutive relation. Under some conditions ice undergoes large strains prior to rupture (>50%), so our relation must be valid for large strains. Ultimately the relation must be three dimensional. To be physically reasonable, the three dimensional relation must remain true under any coordinate transformation and must satisfy the physical principal of objectivity, that is, it must appear the same to any observer.

Taking all of the above into account, as well as including some considerations not covered here, a three dimensional form of the constitutive relations was developed. This will be given below. Here, we give the one dimensional reduction:

$$(1-e)^3 \bar{\sigma} + \hat{\Psi}(1-e)^2 \left[(1-e)\dot{\bar{\sigma}} - 2\dot{e}\bar{\sigma}\right] = \frac{3}{2} \hat{\Phi}(1-e)\dot{e} + \hat{\omega} e \quad (2)$$

where $\bar{\sigma}$ and e are engineering stress and strain - that is, they are referred to the undeformed reference state. The coefficients $\hat{\Psi}$, $\hat{\Phi}$, and $\hat{\omega}$ are the one dimensional restriction of the response coefficients. These are material properties, and in general will depend on stress and strain. The term multiplying $\hat{\Psi}$ is the form of the stress rate term appropriate for large deformations.

It is convenient to rewrite equation 2 as

$$\dot{\bar{\sigma}} = \hat{E}(\bar{\sigma}, e)[\dot{e} - \hat{F}(\bar{\sigma}, e)] \qquad (3)$$

where

$$\hat{E}(\bar{\sigma}, e) = \frac{3\hat{\Phi}}{2(1-e)^2\hat{\Psi}} + \frac{2\bar{\sigma}}{1-e} \qquad (4)$$

and

$$\hat{F}(\bar{\sigma}, e) = \frac{(1-e)^3\bar{\sigma} - \hat{\omega}e}{\frac{3}{2}\hat{\Phi}(1-e) + 2\hat{\Psi}\bar{\sigma}(1-e)^2} \qquad (5)$$

Physically, $\hat{E}$ may be thought of as a Young's modulus, and $\hat{F}$ is simply $\hat{E}$ as measured in a constant load test.

There are four more or less standard tests that can be used to determine mechanical properties. These are: constant strain; constant load; constant strain rate; and constant load rate. These have been analysed for the assumed constitutive relation given by equation 2. Two significant conclusions were reached:

Conclusion 1. Exactly two of these four types are required to determine $\hat{E}$ and $\hat{F}$. Any two will do, but there must be at least two, and a third will add no new information (except, possibly as a consistancy check).

Conclusion 2. It is not possible, through any combination of the four tests described above, to learn anything more about the response of ice beyond $\hat{E}$ and $\hat{F}$. That is, it is not possible, through uniaxial tests alone, to determine separately $\hat{\Psi}$, $\hat{\Phi}$, and $\hat{\omega}$. Thus multiaxial tests are essential. Note that, if one were only interested in uniaxial loading, it would be sufficient to know only $\hat{E}$ and $\hat{F}$.

In view of Conclusion 2, Morland considered the question of whether there might be plausible restrictions on the form of $\hat{\Psi}$, $\hat{\Phi}$, and $\hat{\omega}$ which might permit us to determine them from uniaxial tests. Briefly, he found that setting any one equal to zero leads to physically unacceptable results. However, if we require the ratio $\hat{\omega}/\hat{\Phi}$ to be independent of $\bar{\sigma}$, all three functions can be determined if, in addition to two of the tests described above, we do a relaxation test in which we measure strain as a function of time after a load is removed. The further restriction that

$$\hat{\Phi}(\bar{\sigma}, e) = \hat{\Phi}_\sigma(\bar{\sigma})\,\hat{\Phi}_e(e) \qquad (6)$$

still appears consistent with our limited knowledge of ice behavior, and, in fact, the still further restriction $\hat{E} = \hat{E}_0$ = constant may be satisfactory.

The Three Dimensional Equation

The proposed "simplest" constitutive relation is

$$\underset{\sim}{S} + \Psi\left[\underset{\sim}{S}^{(1)} - \frac{2}{3}tr(\underset{\sim}{S}\,\underset{\sim}{D})\underset{\sim}{1}\right] = \phi_1\underset{\sim}{D} + \phi_2\left[\underset{\sim}{D}^2 - \frac{2}{3}tr(\underset{\sim}{D}^2)\underset{\sim}{1}\right]$$
$$+ \omega_1\left[\underset{\sim}{B} - \frac{1}{3}tr(\underset{\sim}{B})\underset{\sim}{1}\right] + \omega_2\left[\underset{\sim}{B}^2 - \frac{1}{3}tr(\underset{\sim}{B}^2)\underset{\sim}{1}\right] \qquad (7)$$

We will not attempt a complete discussion of this equation here, but some explanation is in order.

The trace of any tensor $\underset{\sim}{A}$, $tr(\underset{\sim}{A})$, is the sum of its diagonal elements. It is one of the three invariants of $\underset{\sim}{A}$, the other two being $tr(\underset{\sim}{A}^2)$ and $\det(\underset{\sim}{A})$. Pressure, p, is

$$p = 1/3tr(\underset{\sim}{\sigma}) \qquad (8)$$

where $\underset{\sim}{\sigma}$ is the stress tensor. Incompressability implies p has no effect. We therefore use the deviatoric stress tensor, $\underset{\sim}{S}$, defined as

$$\underset{\sim}{S} = \underset{\sim}{\sigma} - 1/3tr(\underset{\sim}{\sigma})\underset{\sim}{1} \qquad (9)$$

where $\underset{\sim}{1}$ is the unit tensor. The constant volume requirement implies

$$tr(\underset{\sim}{D}) = \underset{\sim}{0} \qquad (10)$$

where $\underset{\sim}{D}$ is the rate of strain tensor. If any two principal components of $\underset{\sim}{D}$ are known, the third may be found from equation 10. $\underset{\sim}{B}$ is the strain tensor, and $\underset{\sim}{S}^{(1)}$ is the deviatoric stress rate tensor given by

$$\underset{\sim}{S}^{(1)} = \dot{\underset{\sim}{S}} + \underset{\sim}{S}(\underset{\sim}{D} + \underset{\sim}{W}) + (\underset{\sim}{D} - \underset{\sim}{W})\underset{\sim}{S} \qquad (11)$$

where $\underset{\sim}{W}$ is the rotation tensor ($=\underset{\sim}{0}$ for our purposes). This more complicated form of the derivative is the source of the coefficient of Ψ in equation 2 and is a consequence of retaining large deformations. The parameters Ψ, ϕ_1, ϕ_2, ω_1 and ω_2 are response coefficients, and in general depend on the invariants of the tensors $\underset{\sim}{S}$, $\underset{\sim}{D}$, and $\underset{\sim}{B}$, denoted respectively by J_i, I_i, K_i. For an incompressable material there is no dependence on I_1, J_1, or K_3.

One way to view equation 7 is to suppose that we began with a general relation depending linearly on the stress, stress rate, strain and strain rate tensors, and on the square of both the strain and strain rate tensors. This general tensor equation is equivalent to three independent scaler equations. These can be re-arranged so that one equation is the average of the three original equations, and the other two are for departures from the average. The equation for the averages will then relate pressure to changes in volume. Incompressability implies no change in volume and no pressure effects, so this equation is identically satisfied. Thus there are only two independent equations. We remark that since the response coefficients may depend on the invariants of $\underset{\sim}{S}$, equation 7 does not imply only linear dependence on stress.

Spring and Morland (3) argue that $\phi_2 = 0$, and we assume this in the following. Strictly speaking, for a simplest relation we should take $\omega_2 = \theta$, but, following Morland, we retain it here.

The two scaler equations equivalent to equation 7 are:

$$
\begin{aligned}
s_{11} + \Psi[\dot{s}_{11} + \tfrac{2}{3}(s_{11} d_{11} - s_{22} d_{11} - s_{11} d_{22} - 2 s_{22} d_{11}] &= \phi_1 d_{11} \\
&+ \tfrac{1}{3}\omega_1[2\lambda_1^2 - \lambda_2^2 - \lambda_1^{-2}\lambda_2^{-2}] + \tfrac{1}{3}\omega_2[2\lambda_1^4 - \lambda_2^4 - \lambda_1^{-4}\lambda_2^{-4}] \\
s_{22} + \Psi[\dot{s}_{22} + \tfrac{2}{3}(s_{22} d_{22} - s_{22} d_{11} - s_{11} d_{22} - 2 s_{11} d_{11})] &= \phi_1 d_{22} \\
&+ \tfrac{1}{3}\omega_1[2\lambda_2^2 - \lambda_1^2 - \lambda_1^{-2}\lambda_2^{-2}] + \tfrac{1}{3}\omega_2[2\lambda_2^4 - \lambda_1^4 - \lambda_1^{-4}\lambda_2^{-4}]
\end{aligned}
\tag{12}
$$

where s_{ii}, d_{ii}, and λ_i are components of $\underset{\sim}{S}$, $\underset{\sim}{D}$, and $\underset{\sim}{B}$.

An objective of the present project is to determine the functional dependence of Ψ, ϕ_1, ω_1, and ω_2 on the tensor invariants I_i, J_i, and K_i.

There are four test configurations which appear feasible for use in determining these. The stress conditions in these tests are listed in Table 1, with the non-diagonal stress terms being 0. The tests with true triaxial stress (TS) and biaxial stress (BS) are described by equations 12. These are two independent equations describing the response in two directions. They are sufficient in principal to determine the functional dependence of the response coefficients on each invariant separately, although as Morland shows, they do not necessarily determine the dependence over the full range of the invariants.

For conventional triaxial, here called transversely isotropic stress (TIS), and uniaxial stress (US), each of equations 12 reduces to the same equation:

$$
s_{11} + \Psi(\dot{s}_{11} + s_{11} d_{11}) = \phi_1 d_{11} + \tfrac{2}{3}\omega_1(\lambda_1^2 - \lambda_1^{-1}) + \tfrac{2}{3}\omega_2(\lambda_1^4 - \lambda_1^{-2}) \tag{13}
$$

This is a consequence of incompressability, which, in this test, implies that the radial deformation is completely determined by the axial deformation. Nothing can be learned from this test that cannot be learned from the simpler uniaxial test. It is important to note that this is not true if the material is compressible.

In order to recover equation 2 from equation 13, we set $(1-e) = \lambda$,

$$
\hat{\omega} e = \omega_1[1-(1-e)^3](1-e) + \omega_2[1-(1-e)^6], \quad \text{and} \quad \sigma = s_{11} = \overline{\sigma}(1-e).
$$

If we substitute the stress conditions for TIS from Table 1 into the expressions for J_2 and J_3 we get

$$J_2 = \frac{1}{3}(\sigma_{11} - \sigma_{22})^2 \tag{14}$$

$$J_3 = \frac{2}{27}(\sigma_{11} - \sigma_{22})^3 = 2\left(\frac{J_2}{3}\right)^{3/2} \tag{15}$$

Thus for the TIS test J_3 can be written explicitly as a function only of J_2. It is therefore not possible to distinguish dependence on J_3 from dependence on J_2 using TIS data. This is a limitation on the test, and does not depend on the incompressability assumption. It is one reason other tests, such as the BS test, are needed.

Pressure Dependence

Recall that volume changes and pressure were subtracted from equation 7 since incompressability implies no volume change, and no pressure effects. For the same reason we assumed that none of the response coefficients depended upon them. This suggests a simple way to incorporate compressability. First, assume that the response coefficients may depend on J_1, I_1, K_3. Second, add an equation relating pressure and volumne change. Recall that

$$I_1 = d_{11} + d_{22} + d_{33} \tag{16}$$

Thus, I_1 is a measure of the rate of volume change. A plausible equation for I_1 is

$$I_1 = k_1 \dot{J}_1 + k_2 \dot{J}_2 + k_3 \dot{J}_3 \tag{17}$$

We note that while the TIS test is still not able to determine completely the response functions, it does provide useful information if compressability is important.

Summary

A "simplest", three dimensional constitutive relation for sea ice has been described. A study of this equation has shown that:

1. At least two types of uniaxial tests are required to determine uniaxial behavior.

2. In general, behavior of ice under multi-axial loading cannot be inferred from uniaxial tests.

3. The conventional triaxial test has inherent limitations which mean that it cannot be used to completely determine the dependence of the response coefficients on all of the invariants.

4. Thus, a complete determination of the constitutive relations requires at least the use of biaxial or true triaxial tests.

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TABLE 1

Standard Test Configurations for Mechanical Properties Determination.

TEST	ABBREVIATIONS	PRINCIPAL STRESSES
Triaxial Stress	TS	$\sigma_{11} \neq \sigma_{22} \neq \sigma_{33}$
Transversely Isotropic Stress	TIS	$\sigma_{11} \neq \sigma_{22}$, $\sigma_{22} = \sigma_{33} \neq 0$
Biaxial Stress	BS	$\sigma_{11} \neq \sigma_{22}$, $\sigma_{33} = 0$
Uniaxial Stress	US	$\sigma_{11} \neq 0$, $\sigma_{22} = \sigma_{33} = 0$

Technical Memorandum

TO: Dr. E.N. Earle, Shell Development Company
FROM: Dr. M. Mellor, CRREL
DATE: October, 1981

FRACTURE TOUGHNESS MEASUREMENTS FOR ICE

As part of the current sea ice study, CRREL has been asked to comment on fracture toughness measurements for ice and, in particular, to suggest what emphasis should be given to fracture toughness studies.

In responding to the Shell request, we first review some of the relevant terminology and historical developments of the subject.

Toughness. In everyday speech, toughness is a quality which enables people, objects, or materials to endure punishment or strain without yielding. In engineering, toughness is a rather poorly defined concept, but traditionally it has been associated with the capacity of a material to absorb energy before fracturing. Clearly energy alone, as represented by the area under a stress/strain curve, is not an adequate measure of toughness, since high strength and small failure strain could indicate large energy for a very brittle material. Perhaps the best way to define and measure toughness is in terms of the ability to dissipate energy before fracturing. In other words, toughness can be associated with the integral of stress multiplied by inelastic strain, or with total strain energy minus the recoverable strain energy. However, in recent years the term toughness has been appropriated, or perhaps mis-appropriated, for virtually exclusive use within the context of fracture mechanics.

In fracture mechanics the term toughness, or fracture toughness, is defined properly as the critical value of the Irwin parameter G, which is also known as a "crack extension force". G, or G_c, has the dimensions of energy per unit area and it is, in fact, equal to twice the effective specific surface energy for fracture.

However, many practitioners of fracture mechanics refer to the critical stress intensity factor K_c as the fracture toughness of a material, even though K_c has dimensions which have no direct relation to any reasonable definition of toughness. In order to keep things clear, we have to refer back to the origins of fracture mechanics, and to the development of modern notions about fracture toughness.

Griffith Theory. Starting from the observation that the bulk strength of brittle solids is, in general, orders of magnitude lower than the theoretical strength deduced from consideration of interatomic force, A.A. Griffith postulated the existence of minute cracks and associated stress concentrations. Drawing upon the stress analysis given by Inglis for a two-dimensional elliptic crack in an elastic plate, Griffith equated the change of potential energy in the plate to the change of surface energy in the crack as the crack grew in length. For a thin elliptic crack of length 2c:

$$\frac{\partial}{\partial c}\left(\frac{\pi \sigma^2 c^2}{E}\right) = \frac{\partial}{\partial c}\left(4c\gamma\right) \tag{1}$$

where E is Young's modulus for the plate, γ is the specific surface energy of the material and σ is the applied stress (tensile and perpendicular to the long axis of the crack) at which crack growth occurs. Thus

$$\sigma = \left(\frac{2}{\pi}\right)^{1/2}\left(\frac{E\gamma}{c}\right)^{1/2} \tag{2}$$

for plane stress. For plane strain the corresponding relation is

$$\sigma = \left[\frac{2}{\pi(1-\nu^2)}\right]^{1/2}\left(\frac{E\gamma}{c}\right)^{1/2} \tag{3}$$

Where ν is Poisson's ratio. Numerically, the two equations are not much different. Much the same result is obtained by direct consideration of theoretical material strength and stress concentration at the end of an elliptic crack. From consideration of interatomic forces as a function of separation, the theoretical tensile strength of the material σ_* is

$$\sigma_* = \left(\frac{E\gamma}{a}\right)^{1/2} \tag{4}$$

where a is the atomic spacing in the unstrained state. From the Inglis stress analysis for an elliptic crack with tip radius ρ, the stress at the crack tip σ_{ct} is

$$\sigma_{ct} = 2\sigma\left(\frac{c}{\rho}\right)^{1/2} \tag{5}$$

where σ is the applied stress in the plate. Equating σ_{ct} to σ_* for crack growth:

$$\sigma = \left(\frac{\rho}{4a}\right)^{1/2}\left(\frac{E\gamma}{c}\right)^{1/2} \tag{6}$$

in which ρ is considered to be of the same order of magnitude as a for a sharp crack. This version is identical to Griffith's plane stress relation if

$$\rho = (8/\pi)\, a = 2.55\, a \tag{7}$$

In addition to providing a reasonable physical explanation for the discrepancy between theoretical strength and actual strength, Griffith was able to develop a failure criterion for the onset of brittle fracture in multiaxial stress states.

Modification of Griffith Theory. Griffith developed his theory primarily to explain the properties of glass, and the theory was later believed to be generally applicable to brittle solids. However, if the strength equations which contain the surface energy γ are applied to metals or polymers, the predicted strength often turns out to be very much lower than the actual strength of the real material. This can be explained by plastic yielding at critical stress concentrations, which has the effect of blunting the cracks.

In the late forties, Orowan and Irwin independently modified the Griffith equation for strength by taking into account the energy dissipated in localized plastic yielding, while at the same time retaining the elastic analysis for the overall effect of a crack because the plastic yield zones were considered small relative to the crack length. Orowan substituted for the surface energy γ a term which included a specific energy for plastic working γ_p:

$$\sigma = \left(\frac{2}{\pi}\right)^{1/2} \left[\frac{E(\gamma + \gamma_P)}{c}\right]^{1/2} \approx \left(\frac{2}{\pi}\right)^{1/2} \left[\frac{E\gamma_P}{c}\right]^{1/2} \tag{8}$$

for plane stress. The approximation follows from the fact that $\gamma_P \gg \gamma$. Irwin expressed the same idea by denoting the critical rate of change of energy with crack length by a parameter G_c. Being an energy per unit area, G_c has the dimensions of force per unit length, and it is referred to as a crack extension force. In the Irwin formulation

$$\sigma = \left(\frac{1}{\pi}\right)^{1/2} \left(\frac{EG_c}{c}\right)^{1/2} \tag{9}$$

for plane stress. Thus the Orowan and Irwin expressions are identical with

$$G_c = 2(\gamma + \gamma_P) \tag{10}$$

An important feature of the Irwin and Orowan modifications is the combination of elastic and plastic assumptions. The local stress field near a crack tip is allowed to create plastic yielding, but the overall solid matrix is still assumed to behave elastically. Obviously, these assumptions can only be justified if: (1) the spacing between cracks is significantly greater than the extent of the plastic yield zones at the crack tips, and (2) the solid matrix really is elastic.

Fracture Mechanics and Fracture Toughness. The name "fracture mechanics" has come to be used, somewhat restrictively, for study of the effect of cracks on the bulk strength of solid materials. It derives from Griffith theory, and from the later modifications of that theory by Irwin and Orowan, as outlined above.

Griffith's original idea was that fracture occurred when a crack extended without limit because an increment of crack extension involved a gain of surface energy U_s less than the drop of potential energy of the surrounding elastic material U_p:

$$|\delta U_P| > |\delta U_s| \tag{11}$$

Irwin and Orowan introduced the idea of energy dissipation by plastic yielding at a crack tip (δW_p) and the possibility of external work input to the system (δW_e), making the critical energy balance:

$$\delta U_p + \delta W_e \geqslant \delta U_s + \delta W_p \qquad (12)$$

Since $\delta W_p \gg \delta U_s$ and δW_e is, for all practical purposes, zero, the condition simplified to

$$\delta U_p \geqslant \delta W_p \qquad (13)$$

The change of potential energy δU_p as the crack extends by an increment of length δx can be equated to a unit force G multiplied by δx :

$$\delta U_p = G\,\delta x \qquad (14)$$

or

$$G = \frac{\delta U_p}{\delta x} \qquad (15)$$

This is Irwin's crack extension force, which was mentioned earlier. From elastic analysis, the critical value G_c for unstable crack extension is

$$G_c = \frac{\pi c}{E}\,\sigma^2 \qquad (16)$$

for plane stress, and

$$G_c = \frac{\pi(1-\nu^2)c}{E}\,\sigma^2 \qquad (17)$$

for plane strain, where σ is the applied stress at failure.

Analysis of the stress distribution around an idealized crack in an elastic plate gives stress fields that are geometrically similar for geometrically similar "cracks". The absolute magnitude of a given stress component is proportional to the stress applied to the plate, σ, and it is also proportional to the square root of a characteristic linear dimension of the crack, such as the half-length of the major axis c. Thus the effects of geometric scale and stress level can be expressed by a stress intensity factor K which contains the product $\sigma_x \sqrt{c}$. For convenience, K is defined as

$$K = \sigma (\pi c)^{1/2} \tag{18}$$

This is obviously another way of expressing the crack extension force G. In terms of the critical values for failure, K_c and G_c:

$$G_c = \frac{K_c^2}{E} \tag{19}$$

for plane stress, and

$$G_c = \frac{K_c^2 (1 - \nu^2)}{E} \tag{20}$$

for plane strain.

In this summary of crack analyses the basic ideas have been developed with reference to the opening or closing of a two dimensional crack in a plate that is under uniaxial tension or compression. However, in fracture mechanics three distinct types of crack motion are recognized (Fig. 1). Mode I is the simple separation considered for the foregoing discussion. Mode II is in-plane shearing displacement, with opposite faces of a flat crack sliding across each other in the direction of the crack's major axis. Mode III involves twisting, and sliding of opposing crack faces in a direction normal to both axes of the two-dimensional crack. As far as materials testing is concerned, interest centers on Mode I, and virtually all test methods are designed to extend cracks according to Mode I. The critical value of the stress intensity factor for Mode I is denoted by the symbol K_{Ic}.

Fig. 1 Displacement and crack propagation modes

Application of Griffith Theory to Ice. There is no reason to believe that basic Griffith theory will have much relevance to the failure of ice at low strain rates. However, for high strain rates ($\gtrsim 10^{-3}\ s^{-1}$ at typical temperatures) there is ample evidence that ice deforms elastically, with a modulus close to the true elastic modulus. Thus before applying modern fracture mechanics, which was developed largely to explain the inapplicability of Griffith theory for certain materials, we should check to see whether Griffith theory might apply to ice under high strain rates.

Equation (2) provides a means of calculating the uniaxial tensile strength of ice σ_T as a function of the controlling flaw size when Young's modulus E and the specific surface energy γ are known. For ice of very low porosity, ($\leq$ 0.01), the true Young's modulus E is 9 to 10 GPa at typical temperatures. For non-saline ice, the vapour/solid specific energy γ is approximately 0.1 J/m^2, and the liquid/solid surface energy is about 30% of the vapour/solid value (Fletcher, 1970; Hobbs, 1974.* The vapour/solid value is probably the appropriate one

*Liu and Miller (1979) use values that are off by two orders of magnitude due to incorrect conversion of Fletcher's values.

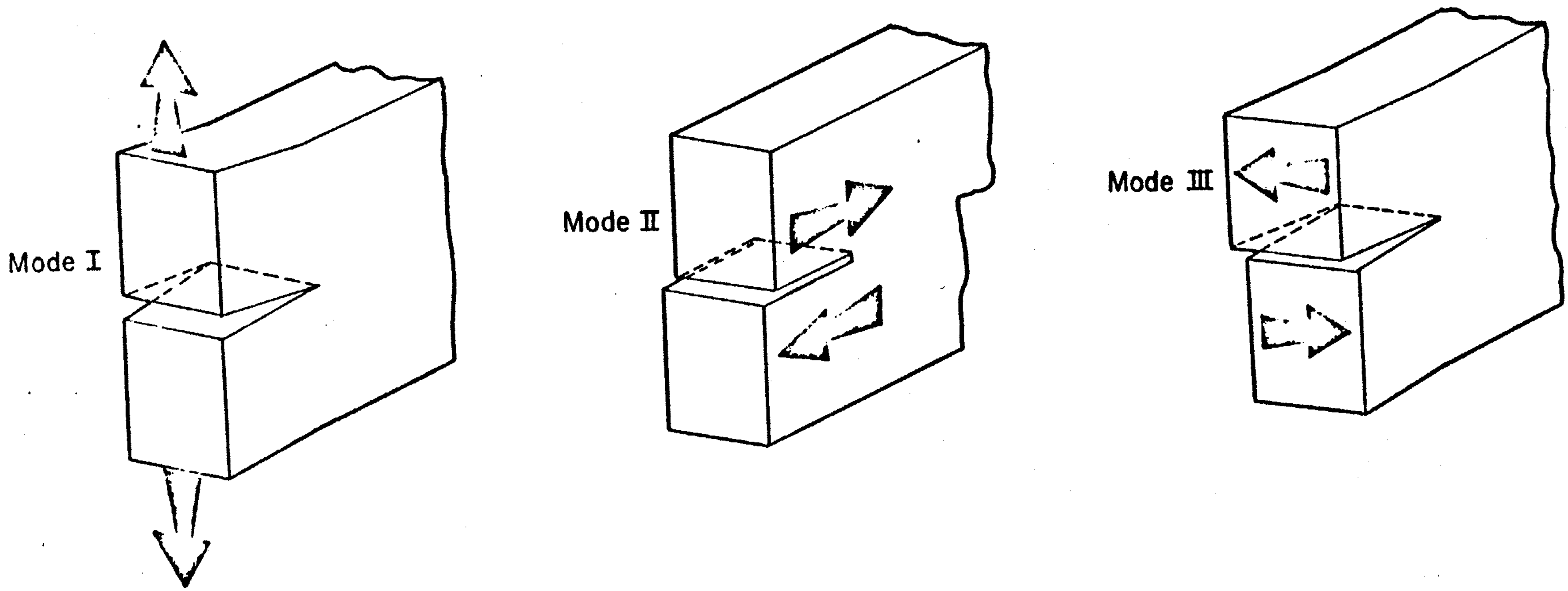

Fig.1. Displacement and crack propagation modes

for consideration of brittle fracture in "cold" ice, but the lower liquid/solid value might be applicable in ice which has a "liquid-like layer" or liquid-filled flaws. The latter condition might give something equivalent to the Rehbinder, or Joffe, effect, whereby γ is reduced by adsorption of certain surface-active chemicals and strength decreases in consequence.

If we substitute into equation (2) $E = 10\ \mathrm{GPa}$ and $\gamma = 0.1\ \mathrm{J/m^2}$,

$$\sigma_T = \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{10^{10} \times 10^{-1}}{c}\right)^{1/2}$$

$$= \frac{2.52 \times 10^{4}}{\sqrt{c}}\ \mathrm{Pa}$$

where the half-length of the controlling flaw, c, is in metres. In figure 2, the resulting calculated values of σ_T are given as a function of the flaw size 2c.

Figure 2 gives a comparison of calculated values with measured values of for non-saline ice, making certain assumptions about "flaw size" for the various test specimens. In none of the test specimens were Griffith cracks actually observed or measured, and so identifiable structural dimensions such as grain size and bubble size have been used to permit plotting of the data. It seems unlikely that the controlling "Griffith crack" could be larger than the grain size in these intact lab specimens, but it is conceivable that the controlling flaws could be smaller than the grain diameter, perhaps by a factor of 2 or 3 if we are considering a mosaic of equant but angular grains.

The real importance of figure 2 is that it gives theoretical strength values which are credible in comparisons with actual test data. While figure 2 does not prove that simple Griffith theory is valid for ice, it certainly gives little reason for rejecting Griffith theory out of hand. In other words, we have no need to invoke a specific energy for plastic working (γ_P) which is orders of magnitude greater than γ , as is apparently the case for metals.

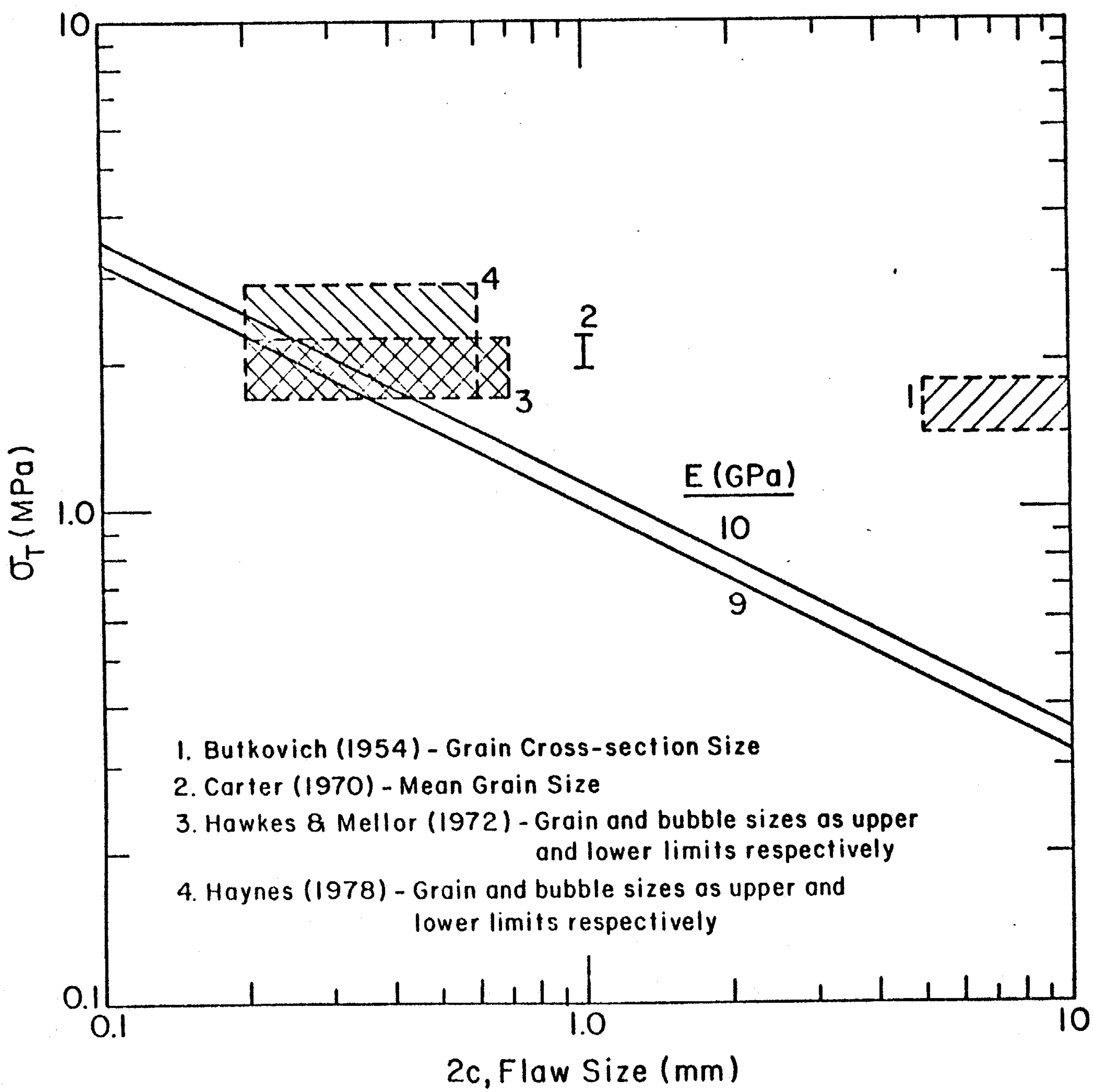

Fig 2 Comparison of theoretical tensile strength with measured values.

If simple Griffith theory were to prove valid for ice, there would be little justification for studying fracture toughness, which is a measure of a material's departure from simple Griffith behaviour. However, various investigations have measured fracture toughness, and it is necessary to review the data.

Fracture toughness of ice. Virtually all fracture toughness measurements on ice depend on tests which flex or pry open a crack in "Mode I". Test data are thus presented in terms of the critical stress intensity factor, K_{Ic}. Because the measured values for K_{Ic} vary greatly, and because we need some intuitive "feel" when considering these values, it is worth recalling what K_{Ic} means.

Toughness is measured by the specific energy dissipation at failure G_c, which is also known as the critical crack extension force. K_{Ic} is related to G_c by

$$K_{Ic} = (E G_c)^{1/2}$$

in plane stress, and

$$K_{Ic} = \left(\frac{E G_c}{1-\nu^2}\right)^{1/2}$$

in plane strain. Thus there is a simple direct relation between K_{Ic} and G_c if E is a constant. K_{Ic} is also related to the overall tensile failure stress of the material σ :

$$K_{Ic} = \sigma (\pi c)^{1/2}$$

where c is the half-length of the controlling cracks. This relation implies that, if c is constant and the stress state does not change, K_{Ic} is directly proportional to the bulk strength of the material.

For ice straining at low rates, say less than 10^{-6} s^{-1}, we would not expect K_{Ic} to have any relevance, since the ice is inelastic and it flows without cracking.

At extremely high rates and low temperatures, ice could conceivably become perfectly elastic and perfectly brittle, and under such conditions the original Griffith theory ought to apply. For such a limit, with γ_P tending to the specific surface energy γ, K_{Ic} would tend to a low value:

$$K_{Ic} \rightarrow (2E\gamma)^{1/2}$$

in plane stress. Taking E = 10 GPa and the grain boundary specific energy $\gamma = 0.1\ J/m^2$ for freshwater ice, the lower limit of K_{Ic} might be about 45 $kN\text{-}m^{-3/2}$ for plane stress.

Measured values of K_{Ic} for ice are typically of order 100 $kN\text{-}m^{-3/2}$. This is not much higher than the "Griffith" value, and it implies that $\gamma_P \approx 5\gamma$, assuming that E is more or less constant.

When strain rate, or loading rate, is varied in a fracture toughness test for a given type of ice, we would expect K_{Ic} to decrease as $\dot{\varepsilon}$ or $\dot{\sigma}$ increases, at least for non-saline ice. While at least one set of experiments shows a trend opposite to this, the overall trend shown by compilation of published data is in the expected sense (Fig. 3, 4). Rate effects were originally expressed in terms of the speed of the testing machine, which is clearly of limited interest, but now the accepted rate variable seems to be $\dot{K}_I$, which is really the inverse of the time to failure. Strain rate has been used as a variable, but there are some problems of interpretation.

In sea ice, K_{Ic} has been found to decrease with increase of loading rate for $\dot{K}_I > 10^{-2}\ kN\text{-}m^{-3/2}\text{-}s^{-1}$, or effective $\dot{\varepsilon} > 10^{-3}\ s^{-1}$ (Urabe et al., 1980; Urabe and Yoshitake, 1981a & b). However, for lower rates K_{Ic} appears to be insensitive to rate (Fig. 5). The lowest measured values for sea ice are lower than the expected "Griffith value" for pure ice.

In discussing rate effects, we have assumed that K_{Ic} will decrease as the material become more elastic and more brittle due to higher loading rates. Extending this line of argument to <u>temperature effects</u>, we might therefore expect K_{Ic} to decrease as temperature decreases, since lower temperature undoubtedly makes ice more elastic and more brittle. However, experimental data (Fig. 6) seem to show exactly the opposite trend, with K_{Ic} increasing as temperature decreases. This observed trend is consistent with the fact that tensile strength σ_T increases as temperature decreases, since K_{Ic} is proportional to strength if the crack length 2c is constant. Nevertheless, there does appear to be fundamental contradiction between the observed temperature effect and the rate effect if the ideas of fracture mechanics are applicable to ice.

If measurements of K_{Ic} are valid, they permit a systematic treatment of flaw size. For a constant value of K_{Ic} and variation of crack length 2c between samples, the tensile strength σ_T might be expected to be inversely proportional to $\sqrt{c}$. Urabe and Yoshitake (1981) tested both notched and un-notched beams with varying grain size in order to calculate flaw size for the ice, and they found a perfect 1:1 correlation between calculated flaw size and observed grain size. However, this experiment appears to merit further discussion, since both σ_{FT} and K_{Ic} were functions of grain size, and the effect of grain size on σ_{FT} appears to be in the wrong direction.

Vaudrey (1977) measured K_{Ic} for sea ice at -10° and $-20^{\circ}C$, and plotted the results against the square root of brine volume for a very narrow range. Vaudrey drew a line on the graph to indicate linear decreas of K_{Ic} with increase in the root of brine volume but, in fact, there was no significant correlation between the variables (K_{Ic} values scattered by a factor of 5). Shapiro et al (1981) made measurements in the same range (brine porosity 0.16 to 0.38), and showed a more convincing decrease of K_{Ic} with increase of porosity, although there was still large scatter.

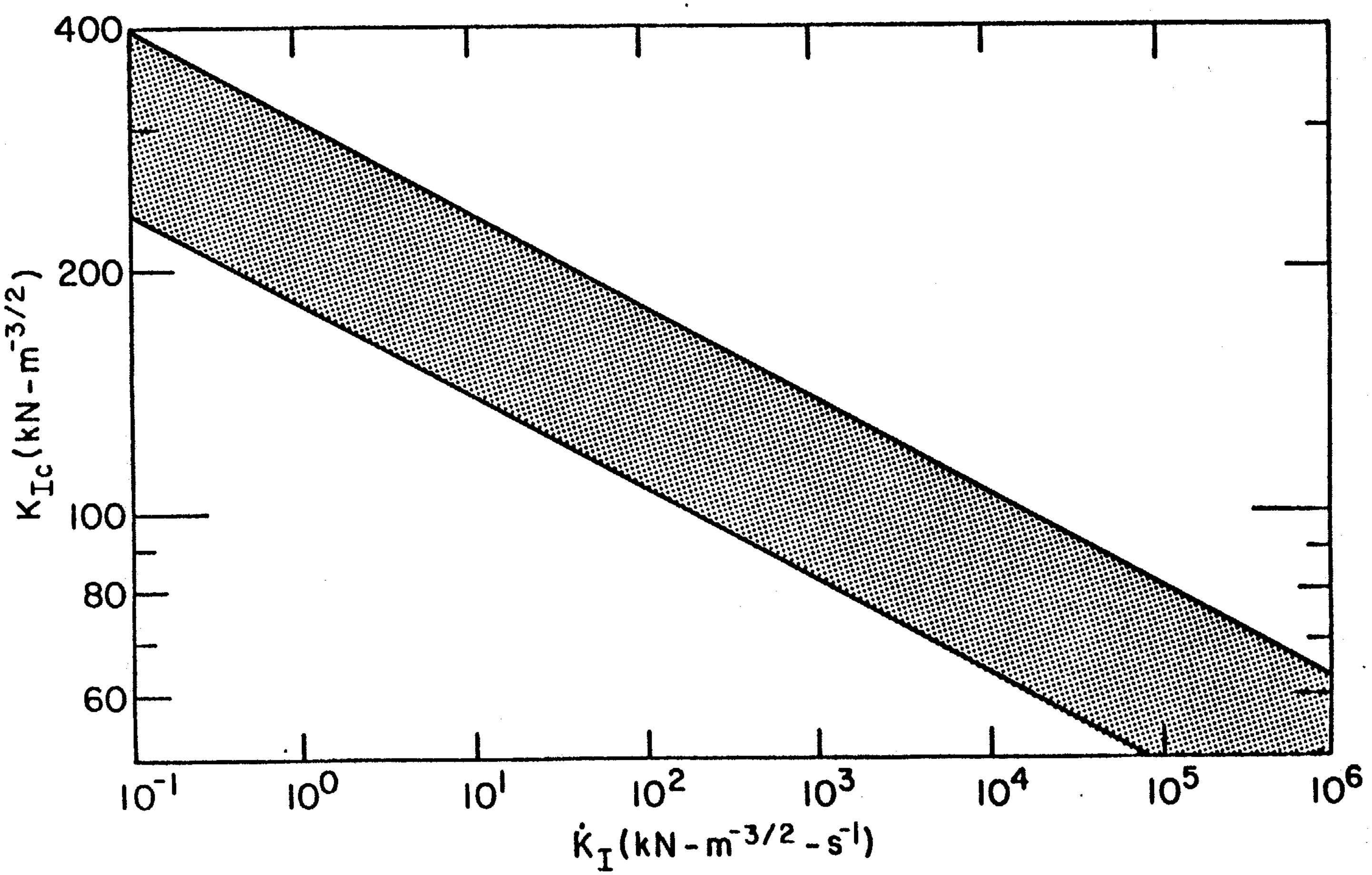

Fig 3. Effect of loading rate on K_{IC} for non-saline ice (from data summarized by Urabe and Yoshitake, 1981).

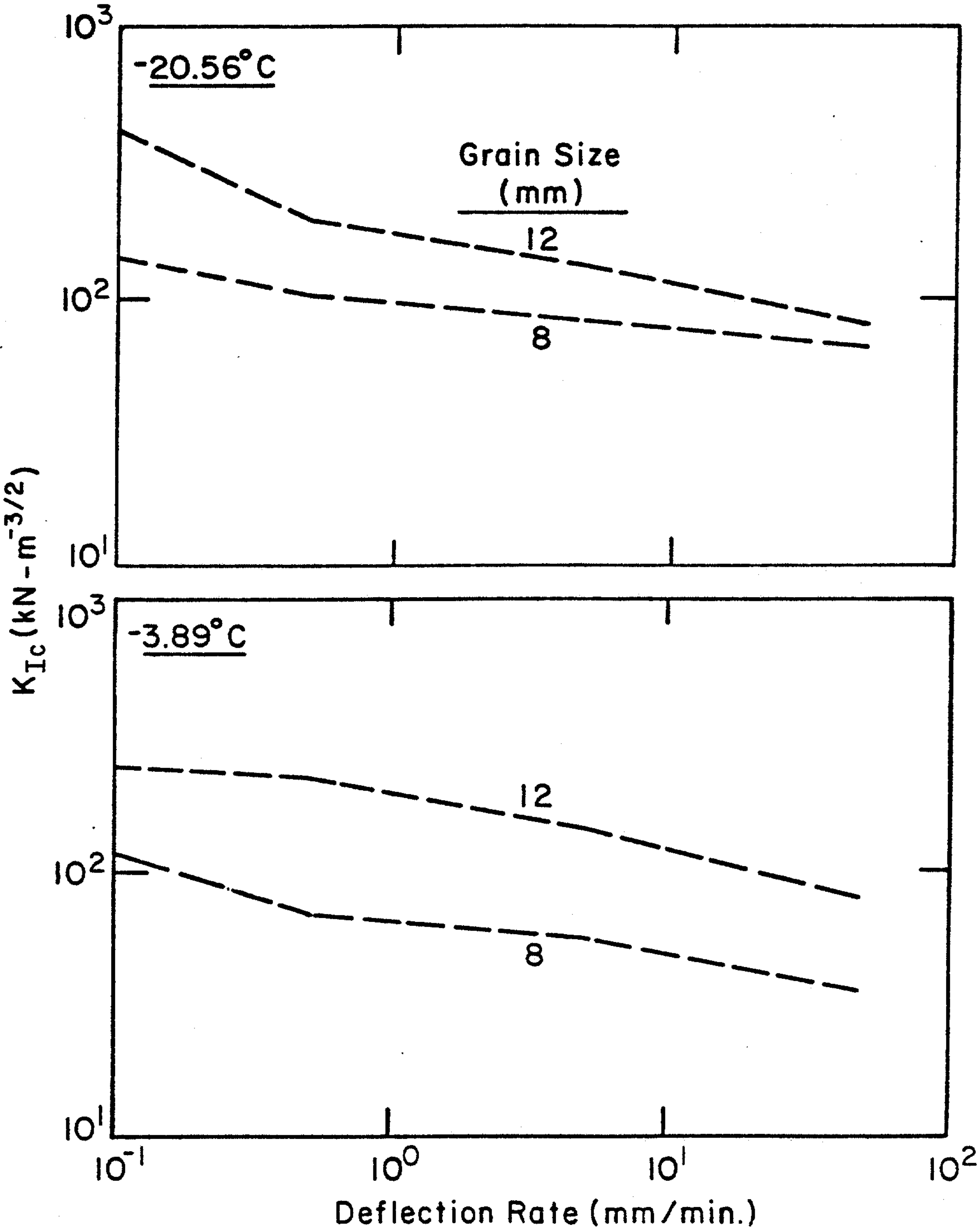

Fig. 4 Variation of K_{IC} with loading rate, grain size and temperature for columnar freshwater ice (Hamza and Muggeridge, 1979).

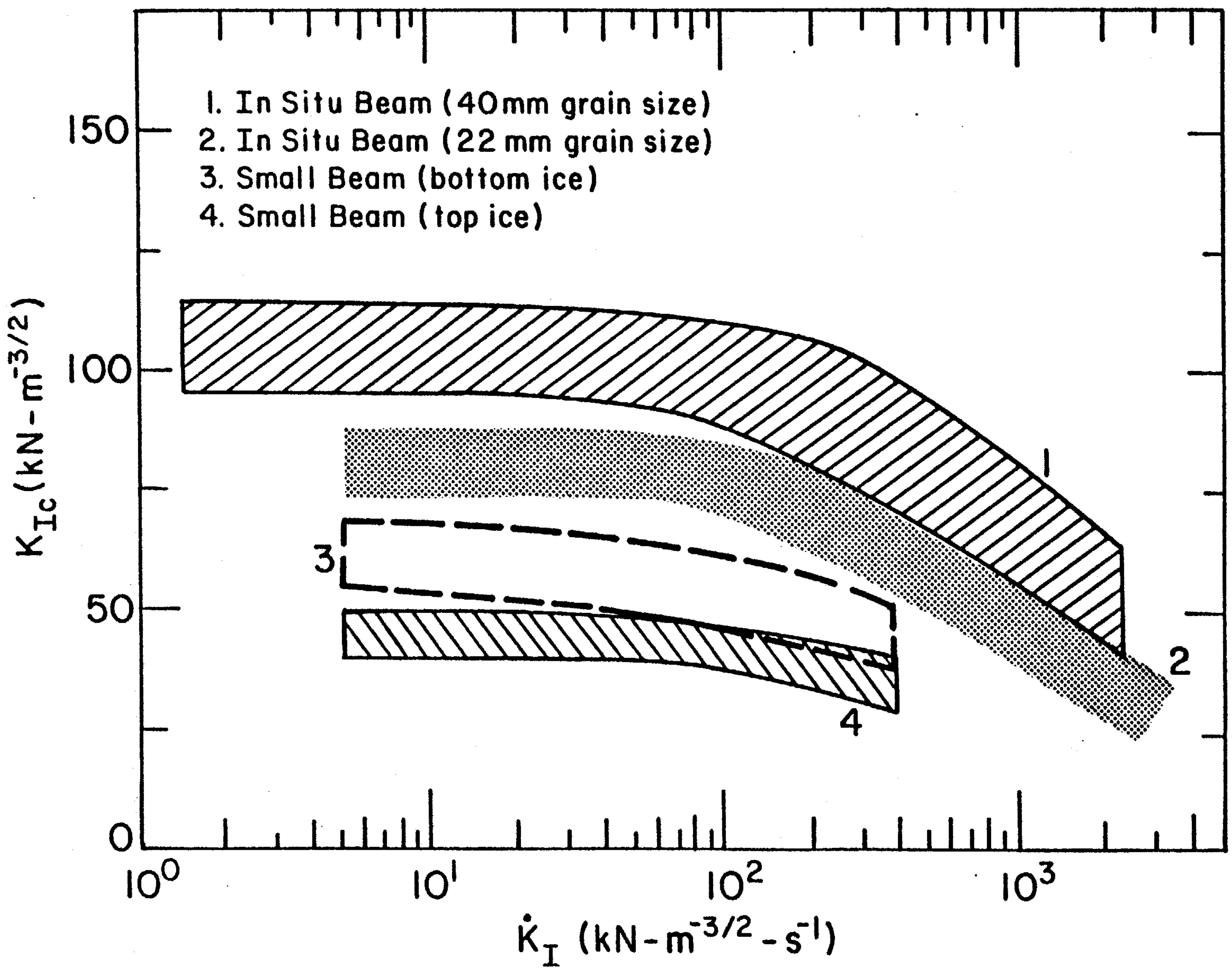

Fig 5 Effect of loading rate on K_{IC} for sea ice (data from Urabe and Yoshitake, 1981).

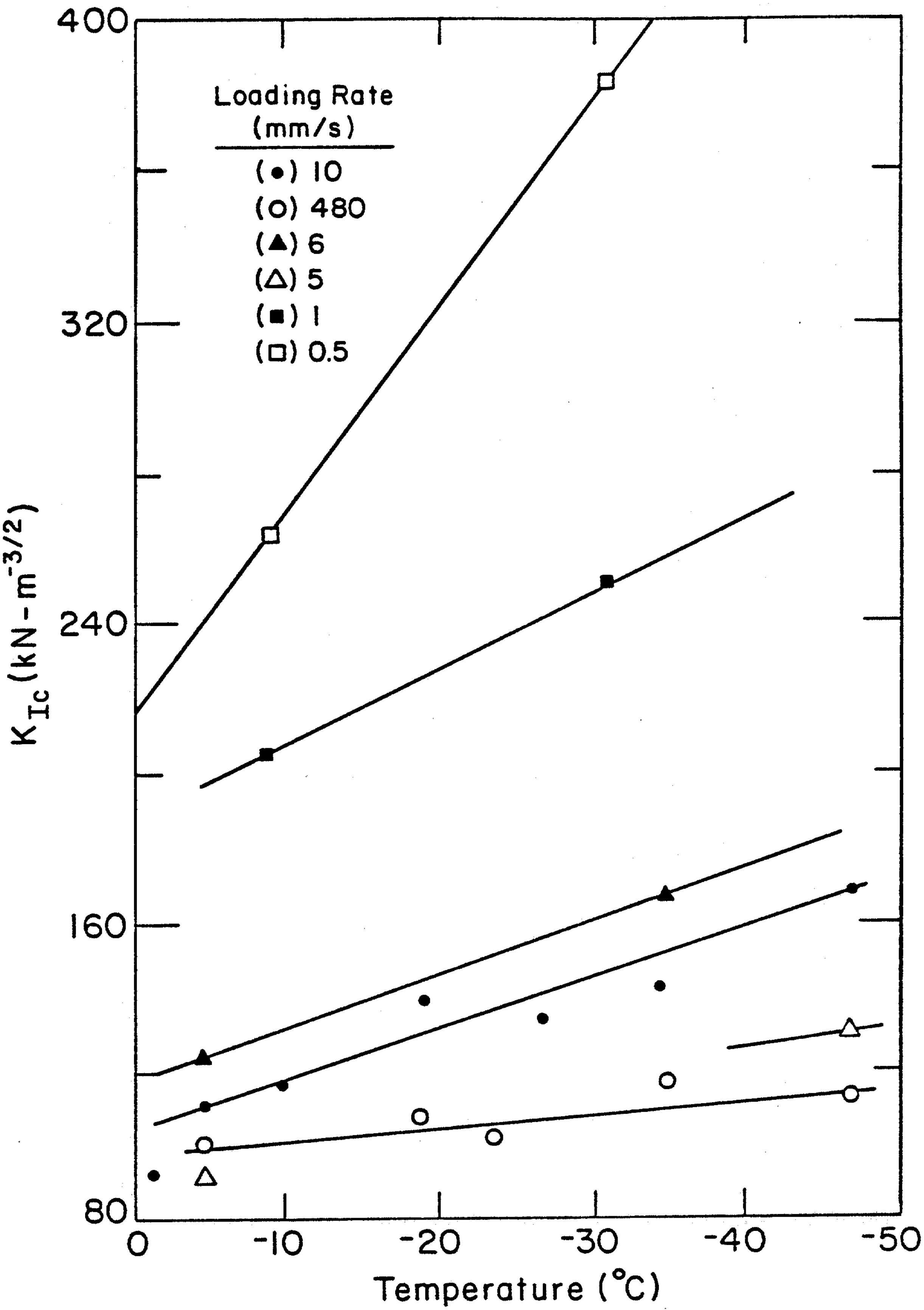

Fig 6 Variation of K_{IC} with temperature and loading rate (from Miller, 1980).

Discussion. When ice is tested at high rates, so that its behaviour is almost purely elastic, simple Griffith theory gives credible predictions of tensile strength, and it predicts values of K_{Ic} which are very close to the values measured at high strain rates.

If ice behaves perfectly elastically, we expect little variation of σ_T and K_{Ic} with temperature, since decrease of temperature involves a slow increase of $\dot{E}$ and a slow decrease of γ . The experimental data for σ_T are consistent with this speech and the data for K_{Ic} given by Miller (fig 6) suggest that the expectation might be borne out if loading rates are sufficiently high.

At the same time, purely elastic behaviour ought to eliminate variation of σ_T and K_{Ic} with strain rate, since neither E nor γ are expected to be significantly rate-dependent. The limited data for σ_T as a function of $\dot{\varepsilon}$ (Hawkes and Mellor, 1972) are consistent with this expectation, but data for K_{Ic} do not appear to be tending to a limiting value for high strain rates.

To sum up the foregoing, ice loaded at high rates behaves elastically, and the limited experimental data for σ_T and K_{Ic} at high rates are not in serious conflict with the predictions of Griffith theory.

Going to the other extreme of behaviour, when ice is strained at very low rates its elastic behaviour is completely overwhelmed by non-linear viscous flow. In this range of behaviour there is no justification for applying elastic fracture mechanics, and K_{Ic} has no significance whatsoever.

This leaves the question of the intermediate range, where elastic deformation and viscous flow both contribute significantly to the total strain. In considering the possible relevance of K_{Ic} for this range, it is important to keep in mind the derivation of the relevant theory, and also the distinction between ideal plasticity and viscous flow.

In deriving the theoretical framework into which K_{Ic} fits, it is assumed that the solid material is elastic-plastic, so that the general matrix can remain elastic while only the most highly stressed zones suffer plastic yielding. However, ice does not have a finite yield stress; it begins to flow at very low stresses, and the flow rate increases with the third or fourth power of stress. Thus, if loading rate is low enough to permit significant inelastic strain prior to final failure, it is unlikely that elastic-plastic fracture mechanics would be applicable. Nevertheless, there might be a range of behaviour, at rates just below those which give purely elastic response, where K_{Ic} is a useful parameter. To examine this possibility, we have to reconsider the experimental data.

K_{Ic} is supposed to be a measure of toughness, and a material's ability to resist weakening by flaws and stress concentrations. We therefore expect K_{Ic} to increase as ductility increases, but we have to keep in mind that increase of K_{Ic} would usually be reflected by an increase of strength.

For ice, K_{Ic} certainly appears to increase as strain rate decreases from the pure elastic range. There are also experimental data showing increase of strength as strain rate decreases through the same range, but these data are not yet conclusive because of the possibility that the trend is caused by imperfections in test technique.

However, as ice temperature decreases, the limited data show K_{Ic} <u>increasing</u>, even though the material is undoubtedly becoming more elastic and more brittle. This trend of K_{Ic} corresponds to the trend shown by strength, indicating a degree of internal consistency in the theoretical ideas, but it is in direct conflict with the strain rate response. Perhaps more to the point, it is in conflict with common sense --- ice does not get tougher as it gets colder.

The reasons for this strange behaviour of K_{Ic} are not immediately obvious, but one might suspect the test method, which is usually beam flexure. Overall, the strength data from beam flexure tests on ice are wildly inconsistent, and it is not hard to see why. The basic assumptions for beam analysis are as follows:

(1) linearly elastic homogeneous material, (2) equal moduli in tension and compression, (3) small strains, with cross-sections remaining plane and mutually parallel. These are met only at very high strain rates, where the test becomes very sensitive to imperfections of specimen preparation and loading technique. Even if a perfect test is made at high rate, fracture initiates at the surface, the zone of critical stress is very thin, and the crack propagates in a stress gradient. If conditions are such that the beam is not perfectly elastic, the degree to which the assumption remain valid varies with temperature and strain rate. Thus the variation of "flexural strength" with strain rate and temperature is unlikely to provide a good indication of the variation of σ_T with strain rate and temperature. When ice beams are notched for fracture toughness tests, a further level of complication is introduced.

Conclusions. When ice behaves elastically ($\dot{\varepsilon} \gtrsim 10^{-3} s^{-1}$ at typical temperatures), simple Griffith theory can be used to assess the effects of flaws and stress concentrations.

When ice is subject to significant creep ($\dot{\varepsilon} \lesssim 10^{-5}$ s^{-1} at typical temperatures), K_{Ic} has no significance.

For the range of behaviour where ice is quasi-brittle (say 10^{-5} to 10^{-3} s^{-1} at typical temperatures), the existing data for K_{Ic} are not easy to accept at face value. Until the apparent contradictions are resolved, it would seem unwise to use K_{Ic} as a design parameter.

Recommendations. Fracture toughness measurements by other research groups should be kept under review. New measurements by Shell probably ought to be deferred until beam flexure testing has been subjected to critical examination. It may be necessary to devise new test methods in order to obtain reliable measurements of K_{Ic} for ice.

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Appendix

Dislocation theory for crack nucleation

Griffith theory and its derivatives deal with the growth of existing cracks. There is another body of theory which deals with the nucleation of cracks by pile-up of dislocations. The latter theory is usually considered to have originated with Zener, and its development is associated with the names Stroh, Petch, Cottrell, Smith and Barnby.

The dislocation theory gives an expression for the effective shear stress τ_{eff} which is needed to produce crack nucleation:

$$\tau_{eff} = \left(\frac{3\pi \gamma G}{8(1-\nu) L} \right)^{1/2} \tag{A-1}$$

where G is the shear modulus and L is the length of the dislocation pile-up. Substituting for G in terms of E and rearranging the equation:

$$\tau_{eff} = \left(\frac{3\pi}{16(1-\nu^2)} \right)^{1/2} \left(\frac{E\gamma}{L} \right)^{1/2} \tag{A-2}$$

This is similar in form to the Griffith equation if L is thought of as a flaw size. However, τ_{eff} is the shear yield stress τ_Y, which is directly proportional to σ_T, minus a "friction stress" τ_i which resists dislocation motion:

$$\tau_{eff} = \tau_Y - \tau_i \tag{A-3}$$

Carter (1970) applied these ideas to ice, taking $\tau_Y = \sigma_T/2$ and L = d/2, where d is the grain diameter (others have taken L = d). He apparently determined τ_i experimentally as 3 kgf/cm^2, but did not explain how this was done. The prediction equation was thus

$$\sigma_T = \left(\frac{3\pi}{2(1-\nu^2)} \right)^{1/2} \left(\frac{E\gamma}{d} \right)^{1/2} + 2\tau_i \tag{A-4}$$

This gives a grain size dependence similar to that predicted by the Hall-Petch relation, instead of the simple $d^{-1/2}$ relation which is obtained by identifying grain size with Griffith cracks.

MECHANICAL PROPERTIES OF SEA ICE

Report M1. Background and Aims

1. Preamble

The meeting with Dr. Earle in Vancouver served to inform me of the overall project and long term aims, and of the present negotiations with CRREL to start a testing programme. My understanding of those aims is as follows:

(i) To obtain accurate data on the time dependent response of sea ice up to stress levels which occur in the engineering applications described below. The time scale of interest can be of order 10^5s, so that primary (decelerating) creep, secondary (a period of steady response or simply an inflexion point) creep, followed by some part of the tertiary (accelerating) creep, can arise under a maintained stress (Mellor, 1980). The rate process is strongly dependent on (deviatoric) stress level, though possible dependence on mean pressure in situations of high confining pressure does not appear to be known. There is also strong dependence on temperature, and the range 273K to 230K is of practical interest. (0 - 43C) A range of deviatoric stress and of confining pressure must be defined. The effects of salinity, up to 0·015, on ductility and anisotropy, particularly of newly formed ice, will be important. Finally, rupture criteria in different failure modes must be determined.

(ii) To construct a constitutive law (or laws) which fits the data to an acceptable degree of accuracy. For application this must describe the response under general stress, and must therefore (a) be expressed in tensor form to be coordinate invariant (b) satisfy the principle of material frame indifference (objectivity - material properties independent of the observer). In addition, any important anisotropy, restriction on material symmetries, must be described, in contrast to the conventional assumption of (full) isotropy. There is also the distinction between compressive and tensile response observed in uni-axial stress which is lost in the very simple tensor model usually assumed in glaciology. It is self-evident that a tensor law to describe general stress response cannot be determined by uni-axial stress tests which, so far, are the only source of accurate data on natural ice. The experimentally difficult combined compression - shear tests of Steinemann (1958) had much more restricted aims and, it seems, were not subsequently refined to improve accuracy. This stress combination would be of great value in distinguishing tensor relation "shapes" which are identical under uni-axial or bi-axial stress (Morland 1979), but is not viewed with enthusiasm by experimenters. Tri-axial stress tests will therefore be essential to determine combined stress features. Rupture criteria must also be formulated.

(iii) To solve boundary-value problems describing the main features of ice-sheet flow, both thin newly formed sheets and thick multi-year sheets, against structures and artificial islands, in order to determine contact pressures and total loads. Contact problems involving both crushing of ice against the structure and bending of the ice sheet prior to breaking are observed, and both situations must be analysed to determine the maximum pressures and loads required in the structure design brief. Both the problem formulation and method of solution can depend on the type of ice law (differential operator, integral operator, fluid, solid) adopted, but since the time scale of the necessary model construction outlined in (i), (ii) must be long (years), it will be necessary to proceed with this applied mathematics development in parallel by considering one or more qualitative laws of anticipated form(s). Solutions may give strong indications of the relative significance of different features of the model in determining structure design parameters, and in turn re-direct the testing programme to concentrate on the more significant features. However, at this stage I would not anticipate any dramatically simplified programme, which should be approached in the spirit outlined in (i). Nevertheless, the solutions can exhibit useful qualitative features arising from the chosen law, with possible magnitude estimates, and may suggest rational approximation procedures to deal with a more complicated law. The problems are highly non-linear and will certainly require a lengthy combination of mathematical analysis and numerical computation. It is important to establish and test procedures to treat some class of problems accurately, to be available as a starting point for subsequent developments, and such analysis will require a lengthy research effort. Since thick ice sheets can have a large temperature variation through the thickness, and creep rate depends strongly on temperature, the mechanical response is significantly non-homogeneous. A constant (mean?) temperature assumption, often used in glaciology, may not yield the correct shape of results, and poses the prior question "what constant temperature?". It does not follow that adopting the worst design parameter from a range of constant temperature solutions is satisfactory, and alternatively this may be far too pessimistic. The most simple approximation procedure to take some account of temperature effects is to assume a temperature profile (based on field data), but at some stage a solution of the coupled momentum - energy balance system is required to test the validity of such approximations.

2. Constitutive models

The non-linearly viscous fluid model adopted in glaciology gives constant strain-rate at constant stress, and neglects entirely the transient primary (and tertiary) creep described by Mellor (1980). In fact, laboratory test time scales at the low deviatoric stresses common in glacier flow are probably much too short to predict the assumed long time 'steady' response, strictly a tertiary creep limit, not secondary creep inflexion, on glaciological time scales. Small strain creep tests in uni-axial stress have been carried out by Sinha (1978), Gold and Sinha (1980) to determine a creep law suitable for engineering applications. The data is fitted to a one-dimensional law for strain in terms of stress and time explicitly, not associated with a tensor relation, and the explicit time dependence violates frame indifference.

A frame indifferent tensor law relating stress to strain-rate and strain acceleration (Morland 1979) shows that the shape of observed transient primary creep followed by a steady secondary creep at constant uni-axial stress can be modelled, but not necessarily the detailed time response at different stresses. A generalisation of this viscoelastic differential operator fluid law (Morland and Spring 1980) has shown that the secondary inflexion followed by tertiary creep to a steady asymptotic limit can also be modelled, with reasonable approximation to an assumed idealised set of creep curves at different stresses. However, to account for the known stress-time behaviour at constant strain-rate, it is necessary to include dependence on stress-rate, and since this term is identically zero in a constant stress test it is never recognised. Thus, constant strain-rate data provides additonal information to constant stress data, and vice-versa. Model fitting to idealised constant stress and constant strain-rate curves has allowed a good description of the time response for one set, but less satisfactory for the other. For example, good time response is modelled for a family of constant stress curves - range of stress levels - but the chosen model has then only flexibility to correlate with a single constant strain-rate curve, and predicts the response at other strain-rates. It would be possible to improve the correlation of this first order differential operator law in stress and strain-rate with given curves by allowing more complex response function dependence, and higher order operators would give more flexibility. However, for coupling with momentum balance laws in applications, the differential operator form, even of first order, is not convenient, and must introduce undesirable features into numerical formulations. This contrasts with homogeneous linear viscoleastic analysis where the constant coefficient differential operators in time allow use of a Laplace transform.

We have also investigated differential models of solid type, incorporating dependence on a reference configuration through stretch rate instead of strain-rate. This dependence is essential to describe anisotropy, and the model will be used to determine uni-axial response following a period of maintained shear, to determine the induced anisotropy from the new sample configuration, even when the model is isotropic with respect to the initial state. In the case of anisotropic newly formed saline ice, it would be possible to start with an initially anisotropic law, and find that the response from later configurations (after confinement by growth) is less anisotropic. Here, though, the decrease of salinity may be the more important factor.

3. Planned investigations

For both fluid and solid type laws I think that an integral operator law can provide better correlation with time response and be more suitable for formulation and numerical treatment of boundary value problems. An investigation of integral laws will be the next stage of my research project supported by US Army ERO in collaboration with CRREL, and, with CRREL approval, developments could be available for use prior to publication. In the meantime, the differential operator fluid law (Morland 1979, Morland and Spring 1980), which is shown to be the most simple differential structure to model known qualitative response of ice in uni-axial constant stress and constant strain-rate tests, can be used to determine minimum test geometries and procedures required to recognise the essential features.

An immediate problem is the shape of the tensor relation describing the viscous fluid behaviour in steady creep at the inflexion point or long time limit (Morland 1979). Since the simplifying assumption that deviatoric stress is parallel to the strain-rate was made in the early days of ice testing, it has not been questioned in glaciology, though never confirmed. It cannot, of course, be assessed by uni-axial stress tests, which can only determine one response function of one argument, but departure from this parallel relation has strong implications for loads involving shear. It has now been shown that bi-axial tests, involving two independent principal stresses, serve only to verify the incompressibility approximation (during creep), and do not separate the two independent response functions of a general frame indifferent (necessarily isotropic) viscous fluid law.

Discussions with Dr. Earle suggested the following immediate investigations to assess further the value of bi-axial load tests and some restricted forms of tri-axial test, illustrated in terms of the viscoelastic fluid models:

(I1) Applications of independent principal stresses $\sigma_1(t)$, $\sigma_2(t) \equiv \sigma_3(t)$, with distinct time dependence to see if any separation of response functions is possible. For example, combinations such as $\sigma_1 = \sigma_o H(t)$, σ_o constant, with $\sigma_2(t)$ continuous (zero at $t = 0$), or vice-versa, to investigate any order of loading effects, or $\sigma_2 = \lambda\sigma_1$ with λ constant, $\sigma_1(t)$ continuous.

(I2) Application of a longitudinal stress with a kinematic constraint, such as zero displacement, in one lateral direction, free displacement in the third direction, or alternatively, application of two independent principal stresses $\sigma_1(t)$, $\sigma_2(t)$, with $\sigma_3(t) = 0$.

(I3) Analysis of tri-axial loading, that is, three independent principal stresses. In particular, to determine the extent of information obtained by a restricted range of tri-axial loads to see what minimal programme of tests is required to describe important combined stress situations.

(I4) Inclusion of the third (odd) strain-rate or stress invariant in response function arguments to distinguish compressive and tensile response, not, as yet, explored in any theoretical models, but of significant practical importance. Each of the investigations (I1) - (I3) should be followed in this manner, noting the difficulty of applying tensile stress in more than one direction, so that the range of the odd invariant will be restricted. We need to know how restricted, and if important combined stress configurations will be omitted from the domain covered. Furthermore, how will dependence on the odd invariant influence simple shear response, which should be directionally invariant.

The above investigations all appear feasible within a modest time scale so that conclusions could influence test programmes in the early stages. It seems sensible to treat the (I4) question simultaneously with (I1), (I2) before starting (I3). Progress usually prompts more pertinent questions so the themes (I1) - (I4) may not be exhaustive. Looking ahead, these sort of questions must be raised and examined for integral operator laws, with experience gained from the differential operators directing attention to the key issues.

5. Rupture

To complete this report I will add a few comments on rupture. Firstly, this term should always be used since failure (in ice mechanics) is ambiguous, certainly used in the sense of peak stress in a constant strain-rate test beyond which the stress relaxes smoothly without any material fracture. The most simple theoretical model is the postulate of a critical value of some given stress combination being reached, necessarily a scalar function of the stress invariants for frame indifference, analogous to the "plastic yield" conditions adopted in rock/soil mechanics (Morland 1971). Here bi-axial tests can investigate rupture over a two-dimensional stress domain, so tentatively separating shear stress and mean pressure influence. This approach is strictly empirical and within the framework of continuum mechanics, since it does not invoke any underlying physics cause, but a correlated dual approach may be useful.

In the recent IUTAM Symposium a number of scientists proposed that strain (not stress) was the essential physical ingredient of a rupture criterion. This contrasts strongly with the model described above, since a given current stress allows a variety of strain histories. This appears to be an area where careful testing and interpretation is required to decide the essential dependence.

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Dr. L. W. Morland
30th September 1980

MECHANICAL PROPERTIES OF SEA ICE

Report M2. Viscoelastic Differential Operator Model and Test Geometries

1. Introduction

Report M1 listed some immediate investigations to assess the values of conventional bi-axial and limited forms of tri-axial tests to determine the tensor shape and coefficient dependence of laws for an incompressible viscoelastic fluid. Specifically, (I1) seeks the significance of distinct time dependences of the principal stress $\sigma_1(t)$, $\sigma_2(t) \equiv \sigma_3(t)$, in a bi-axial test; (I2) proposes the examination of alternative two parameter loading configurations, namely (a) $\sigma_3(t) \equiv 0$, $\sigma_1(t)$, $\sigma_2(t)$ independent, (b) $\sigma_3(t) \equiv 0$, $\sigma_1(t)$ arbitrary, strain-rate $d_2(t) \equiv 0$ (zero displacement in x_2-direction); (I3) is the analysis of tri-axial loading to determine how restricted ranges of the principal stresses describe general stress configurations; and (I4) introduces dependence on two independent stress invariants and two independent strain-rate invariants, necessary to distinguish compressive and tensile response in uni-axial stress.

This Report covers each of the investigations (I1) - (I4) in relation to a differential operator model relating stress, stress-rate, strain-rate, strain-acceleration. It is shown that the bi-axial test can provide no more information than the uni-axial stress test, except that incompatible results would invalidate the adopted model. The analysis is presented for general tri-axial loading, with coefficients depending on all four stress and strain-rate invariants, and the bi-axial configuration $\sigma_2 \equiv \sigma_3$ and alternatives (a), (b), with $\sigma_3 \equiv 0$, are extracted as special cases. It is shown that three independent stresses are not necessary to determine the model, and that alternative (a) provides (in principle) a full cover of stress configurations, including uni-axial tension, even with the restriction $\sigma_1 \leqslant 0$, $\sigma_2 \leqslant 0$ (both applied stresses compressive), and must therefore become a priority in constant stress test programmes. Correlation between model and data requires an analysis of two simultaneous non-linear differential equations, for which the shape of the observed response is significant, and this will be an important part of the theoretical research programme. Constant stress response is not sufficient to determine the mechanical properties, and must be complemented by constant strain-rate response. Here, the more practical alternative (b) with $\sigma_2(t)$ a resulting response, not applied, is only of limited value. This specific investigation arose out of my discussions with Dr. Earle, emphasizing the value of interchange of views.

The full analysis of this differential operator model applied to uni-axial stress response has only been recently completed as part of an ERO contract in collaboration with CRREL, and will appear in an Annual Report. A paper for publication is presently being typed, and I have requested permission to give you an advance copy on completion. Further to the outline in an earlier technical report you have, it is now shown that the entire family of strain-rate v time responses at different constant-stresses, and the family of stress v time responses during primary stress increase at different constant strain-rates, can be correlated with the model. This means that constant strain-rate response contains additional information to constant stress response, but reflects some common properties in that the stress-relaxation is now predicted. Alternative combinations of correlation and prediction may be adopted, and in particular the roles of constant stress and constant strain-rate response may be reversed; full correlation with constant strain-rate response, partial correlation with constant stress response.

I will start by describing the uni-axial response at constant stress and constant strain-rate which the model must predict, and the necessary and sufficient forms of differential relations (up to stress-rate and strain-acceleration). These cannot determine explicitly the corresponding tensor relations for general stress configurations, but a general shape is inferred and particular examples are proposed which could be a starting point for analysis of the simultaneous differential equations corresponding to case (a). This approach to model construction will be relevant when compatibility with two-dimensional observations must be ensured. My presentation here is different from that of the paper which starts from tensor relations.

Simple shear motion is analysed for the general model, showing that normal stresses on planes perpendicular to the shear plane are necessary. More important, the effects on the model coefficients of assuming invariance of the shear stress-shear strain-rate relation under reversal of shear stress direction can be investigated. Even though simple shear tests are not practical, this property appears entirely plausible. It implies no restrictions on the coefficients. Next, tri-axial stress is analysed and specialised for cases (a), (b), deriving the simulaneous differential equations arising from constant stress and constant strain-rate tests. A detailed analysis of the domain of stress invariants covered by compressive loads in case (a) is presented, showing in particular that $\sigma_2 = \lambda\sigma_1$ for different constant $\lambda > 0$, constant $\sigma_1 < 0$, can cover the full domain, including tensile configurations

2. Differential operator model

Figure 1 from Morland and Spring (1980) shows the typical longitudinal compressive strain-rate $r(t)$, where t denotes time, at a constant uni-axial compressive stress σ, and Fig. 2 shows the typical stress $\sigma(t)$ at constant strain-rate r (Mellor 1980). In the primary creep r decreases from the initial strain-rate $r_o(\sigma)$ to a minimum strain-rate $r_m(\sigma)$, which corresponds to the inflexion point (secondary creep) on a strain-time curve, then in the tertiary creep r increases to an asymptotic equilibrium strain-rate $r_e(\sigma)$. Results are stated for $r_e < r_o$, but analogous results can be given for $r_e > r_o$. It is suggested that $r_e \to r_m$ as $\sigma \to 0$, so that tertiary creep becomes indistinct from secondary creep at low stress. Note that the significant non-monotonic strain-rate implies that strain-rates r in $r_m < r < r_e$ occur at two distinct times during the creep. In turn this requires that a differential equation for $r(t)$ which expresses the strain-acceleration $\dot{r}(t)$ as a function of r must have two branches. Similarly, at constant r, stress increases from zero to a maximum $\sigma_M(r)$ at time $t_M(r)$, then relaxes asymptotically to an equilibrium level $\sigma_E(r)$. Consistency of the two types of response for the same material implies the inverse relations

$$r_e^{-1}(r) = \sigma_E(r), \quad r_m^{-1}(r) = \sigma_M(r) \ . \tag{1}$$

We now seek a differential equation for $r(t)$ which describes the response shown in Fig. 1 at constant σ, strictly a family of such curves for different σ. A viscous fluid law which implies constant r at constant σ is quite inappropriate, but inclusion of a strain-acceleration term, and hence dependence on $\dot{r}$ in uni-axial stress, allows this variation with time. A more limited description of primary creep with secondary creep an asymptotic equilibrium limit was so modelled by Morland (1979). The principle of material frame indifference requires that time does not occur explicitly in response coefficients of the model, which must be functions of invariants of the arising tensors. For uni-axial stress, coefficients will reduce to functions of σ and r, so that the differential equation relates $\dot{r}$ and r with σ as a constant parameter. To describe the response shown in Fig. 1 it must yield two branches:

$$\begin{aligned} \dot{r}_-(t) &= R_-(r,\sigma) \leqslant 0, \qquad r_o \geqslant r \geqslant r_m , \\ \dot{r}_+(t) &= R_+(r,\sigma) \geqslant 0, \qquad r_m \leqslant r < r_e , \end{aligned} \tag{2}$$

where $R_-(r_m,\sigma) = R_+(r_m,\sigma) = 0$, $R_+ \to 0$ as $r \to r_e(\sigma)$, and R_-, R_+ are given data. It is convenient to define $R_+(r,\sigma) \equiv 0$ for $r_e < r < r_o$. The appropriate differential equation is

$$\dot{r}^2 + f(r,\sigma)\dot{r} = F(r,\sigma), \quad r(0) = r_o(\sigma), \tag{3}$$

which has two branches

$$2\begin{pmatrix} \dot{r}_- \\ \dot{r}_+ \end{pmatrix} = -f \pm (f^2 + 4F)^{\frac{1}{2}}, \qquad \begin{matrix} \leqslant \\ \geqslant \end{matrix} \; 0 \tag{4}$$

if $F \geqslant 0$, $f \geqslant 0$. Comparing (2) and (4),

$$f = -(R_+ + R_-), \quad (f^2 + 4F)^{\frac{1}{2}} = R_+ - R_-, \quad r_m \leqslant r \leqslant r_o, \tag{5}$$

which determine $f(r,\sigma)$, $F(r,\sigma)$ with $F \equiv 0$ for $r_e \leqslant r \leqslant r_o$. Various properties of f and F required to reproduce the smooth response shown in Fig. 1 are deduced by Morland and Spring (1980).

A similar analysis of the constant strain-rate response $\sigma(t)$ shown in Fig. 2 yields the differential equation

$$\dot{\sigma}^2 - g(r,\sigma)\dot{\sigma} = G(r,\sigma), \quad \sigma(0) = 0, \tag{6}$$

with two branches

$$2\begin{pmatrix} \dot{\sigma}_+ \\ \dot{\sigma}_- \end{pmatrix} = g \pm (g^2 + 4G)^{\frac{1}{2}} . \tag{7}$$

We require $G \geqslant 0$, $g \geqslant 0$, and $G \equiv 0$ for $r \geqslant r_e$ using the inverse relation $(1)_1$, and also $f = F = g = G \equiv 0$ for $r \leqslant r_m$. The various tensor laws relating stress, stress-rate, strain-rate, strain-acceleration, compatible with (3) and (6), deduced by Morland and Spring (1980) all imply

$$F \equiv G, \tag{8}$$

so only one response function $g(r,\sigma)$ is available to correlate (7) with observed data once f, F are determined by (5). For example, the primary stress increase given by $\dot{\sigma}_+$ can be matched by choice of g, then the relaxation $\dot{\sigma}_-$ is predicted. Thus, constant stress and constant strain-rate response are neither fully dependent nor fully independent.

The full correlation (5) requires both f and F to depend on both r and σ, and in turn the tensor coefficients must all depend on at least one strain-rate invariant and one stress invariant. Dependence on one stress component σ and one strain-rate component r cannot distinguish dependence on two independent stress and two independent strain-rate invariants, though some restrictions are imposed by seperate compression and tension results.

A simple form of frame indifferent tensor relation between stress, stress-rate, strain-rate, strain-acceleration, for an incompressible fluid is

$$\begin{aligned} &\psi_1 \underset{\sim}{S} + \psi_3\left[\dot{\underset{\sim}{S}} + \underset{\sim}{S}(\underset{\sim}{D} + \underset{\sim}{W}) + (\underset{\sim}{D} - \underset{\sim}{W})\underset{\sim}{S} - \frac{2}{3}\operatorname{tr}(\underset{\sim}{S}\,\underset{\sim}{D})\underset{\sim}{1}\right] \\ &\quad = \phi_1 \underset{\sim}{D} + \phi_2\left[\underset{\sim}{D}^2 - \frac{2}{3} I_2 \underset{\sim}{1}\right] + \phi_3\left[\dot{\underset{\sim}{D}} + \underset{\sim}{D}\,\underset{\sim}{W} - \underset{\sim}{W}\,\underset{\sim}{D}\right], \end{aligned} \tag{9}$$

where $\underset{\sim}{D} + \underset{\sim}{W}$ is the symmetric-skew decomposition of the spatial velocity gradient, $\underset{\sim}{S}$ is the stress deviator, and the superposed dot denotes material time derivative. Thus

$$\underset{\sim}{S} = \underset{\sim}{\sigma} - \frac{1}{3}\operatorname{tr}\underset{\sim}{\sigma}\,\underset{\sim}{1}, \quad \underset{\sim}{D} + \underset{\sim}{W} = \operatorname{grad}\underline{v}, \quad \operatorname{tr}\underset{\sim}{S} = \operatorname{tr}\underset{\sim}{D} = 0, \tag{10}$$

where $\underset{\sim}{\sigma}$ is the Cauchy stress and $\underline{v}$ the velocity. The form (9) assumes linear dependence on the stress-rate and strain-acceleration tensors. The coefficients can be functions of any invariants of $\underset{\sim}{S}$, $\underset{\sim}{D}$, $\dot{\underset{\sim}{S}}$, $\dot{\underset{\sim}{D}}$ and their products, and to obtain (3) and (6) in uni-axial stress it is necessary to incorporate some dependence on rate invariants. A basic set of invariants is

$$\begin{aligned} &J_2 = \frac{1}{2}\operatorname{tr}\underset{\sim}{S}^2, \quad J_3 = \det\underset{\sim}{S}, \quad I_2 = \frac{1}{2}\operatorname{tr}\underset{\sim}{D}^2, \quad I_3 = \det\underset{\sim}{D}, \\ &\dot{J}_2, \ \dot{J}_3, \ \dot{I}_2, \ \dot{I}_3 . \end{aligned} \tag{11}$$

In uni-axial stress $-\sigma_{11} = \sigma$ (> 0 for compression), other $\sigma_{ij} = 0$, with corresponding strain-rate $-D_{11} = r$, $D_{22} = D_{33} = \frac{1}{2}r$, the constant stress relation is

$$-\frac{2}{3}\psi_1\sigma + \frac{2}{3}\psi_3\sigma r + \phi_1 r - \frac{1}{2}\phi_2 r^2 + \phi_3\dot{r} = 0 , \tag{12}$$

to be compared with (3), and the constant strain-rate relation is

$$-\frac{2}{3}\psi_1\sigma - \frac{2}{3}\psi_3\dot{\sigma} + \frac{2}{3}\psi_3\sigma r + \phi_1 r - \frac{1}{2}\phi_2 r^2 = 0 , \tag{13}$$

to be compared with (6). Both (12) and (13) are special cases of the tri-axial stress relations constructed later. Two examples of the simultaneous comparison (12) with (3) and (13) with (6) are cases in which rate dependence is entirely through the tensors $\dot{\underset{\sim}{S}}$ and $\dot{\underset{\sim}{D}}$, and entirely through the invariants $\dot{J}_2$ and $\dot{I}_2$. First

$$\begin{aligned} &\psi_1 = 1, \ \frac{2}{3}\psi_3 = -\dot{\sigma} + g(r,\sigma) - r\sigma , \quad \phi_3 = \dot{r} + f(r,\sigma) \\ &F(r,\sigma) = G(r,\sigma) = -\phi_1 r + \frac{1}{2}\phi_2 r^2 - r\sigma g(r,\sigma) + r^2\sigma^2 + \frac{2}{3}\sigma , \end{aligned} \tag{14}$$

in which ψ_3, ϕ_3 depend on stress-rate and strain-acceleration invariants respectively as well as on J_2, J_3, I_2, I_3, and ϕ_1, ϕ_2 depend only on the latter set. Second

$$\begin{aligned} &\phi_3 = \psi_3 = 0, \ \frac{2}{3}\psi_1 = -\dot{r}^2 - f(r,\sigma)\dot{r} - \dot{\sigma}^2 + g(r,\sigma)\dot{\sigma} , \\ &F(r,\sigma) = G(r,\sigma) = -\phi_1 r + \frac{1}{2}\phi_2 r^2 , \end{aligned} \tag{15}$$

in which ψ_1 depends on the rate invariants.

3. Simple shear motion

Consider a simple shear motion

$$v_1 = 2\gamma\, x_2 , \quad v_2 = v_3 = 0 , \tag{16}$$

for which

$$\underset{\sim}{D} = \begin{pmatrix} 0 & \gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad \underset{\sim}{W} = \begin{pmatrix} 0 & \gamma & 0 \\ -\gamma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{17}$$

where γ is the shear strain-rate. The corresponding stress has the form

$$\underset{\sim}{\sigma} = \begin{pmatrix} \sigma_1 & \tau & 0 \\ \tau & \sigma_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \underset{\sim}{S} = \begin{pmatrix} \frac{2}{3}\sigma_1 - \tau & \tau & 0 \\ \tau & \frac{2}{3}\sigma_2 - \tau & 0 \\ 0 & 0 & -\frac{1}{3}(\sigma_1+\sigma_2) \end{pmatrix}, \tag{18}$$

in which $\sigma_3 = 0$ is compatible with (9), but in general non-zero normal stresses σ_1 and σ_2 are required. The invariants are

$$J_2 = \tau^2 + \frac{1}{3}(\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2), \; J_3 = \frac{1}{27}(\sigma_1 + \sigma_2)(\sigma_1 - 2\sigma_2)(2\sigma_1 - \sigma_2),$$
$$I_2 = \gamma^2, \quad I_3 = 0, \tag{19}$$

so J_3 is independent of τ and I_3 is zero for all γ ; no I_3 dependence could be detected. The tensor relation (9) yields three independent simultaneous first order differential relations

$$\psi_1(\tfrac{2}{3}\sigma_1 - \tfrac{1}{3}\sigma_2) + \psi_3\left[(\tfrac{2}{3}\dot{\sigma}_1 - \tfrac{1}{3}\dot{\sigma}_2) - \tfrac{4}{3}\tau\gamma\right] = \tfrac{1}{3}\gamma^2(\phi_2 - 6\phi_3),$$
$$\psi_1(\tfrac{2}{3}\sigma_2 - \tfrac{1}{3}\sigma_1) + \psi_3\left[(\tfrac{2}{3}\dot{\sigma}_2 - \tfrac{1}{3}\dot{\sigma}_1) + \tfrac{8}{3}\tau\gamma\right] = \tfrac{1}{3}\gamma^2(\phi_2 + 6\phi_3), \tag{20}$$
$$\psi_1\tau + \psi_3\left[\dot{\tau} + 2(\tfrac{2}{3}\sigma_1 - \tfrac{1}{3}\sigma_2)\gamma\right] = \phi_1\gamma + \phi_3\dot{\gamma}.$$

For given coefficients $\phi_1, \phi_2, \phi_3, \psi_1, \psi_3,$ and prescribed $\gamma(t)$, the solution $\sigma_1(t), \sigma_2(t), \tau(t)$ will not in general have $\sigma_1 \equiv 0$ or $\sigma_2 \equiv 0$, so these normal stresses are required to maintain the simple shear motion. Similarly, an analysis of simple shear stress shows corresponding non-zero axial strain-rate D_{11}, D_{22}. Now if the direction of shear is reversed, $\gamma \to -\gamma$, we can safely assume $\tau \to -\tau$, but σ_1, σ_2 are unchanged, so J_2, J_3, I_2, I_3 are unchanged. Thus ϕ_1, ϕ_3, ψ_1, ψ_3, can be arbitrary functions of I_2, I_3, J_2, J_3, and their rates, consistent with each relation of (20).

4. Tri-axial stress

Consider three independent principal stresses σ_1, σ_2, σ_3, with corresponding deviatoric compenents S_1, S_2, $S_3 = -(S_1 + S_2)$ where $S_j = \sigma_j - \frac{1}{3}\sigma_{kk}$:

$$\underset{\sim}{\sigma} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}, \qquad \underset{\sim}{S} = \begin{pmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & -(S_1 + S_2) \end{pmatrix}$$

$$J_2 = S_1^2 + S_2^2 + S_1 S_2 = \tfrac{1}{3}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - (\sigma_2\sigma_3 + \sigma_3\sigma_1 + \sigma_1\sigma_2)\right],$$

$$J_3 = -S_1 S_2 (S_1 + S_2) = \tfrac{1}{27}\left[(2\sigma_1 - \sigma_2 - \sigma_3)(2\sigma_2 - \sigma_1 - \sigma_3)(2\sigma_3 - \sigma_1 - \sigma_2)\right]. \tag{21}$$

The corresponding strain-rate and rotation are

$$\underset{\sim}{D} = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & -(d_1 + d_2) \end{pmatrix}, \quad \underset{\sim}{W} = \underset{\sim}{0},$$

$$I_2 = d_1^2 + d_2^2 + d_1 d_2, \quad I_3 = -d_1 d_2 (d_1 + d_2) . \tag{22}$$

Now the tensor relation (9) yields two independent relations

$$
\begin{aligned}
&\psi_1 S_1 + \psi_3\left[\dot{S}_1 + \tfrac{2}{3}(S_1 d_1 - S_2 d_1 - S_1 d_2 - 2S_2 d_2)\right] \\
&\qquad = \phi_1 d_1 + \phi_2\left[\tfrac{1}{3}d_1^2 - \tfrac{2}{3}d_2^2 - \tfrac{2}{3}d_1 d_2\right] + \phi_3 \dot{d}_1 , \\
&\psi_1 S_2 + \psi_3\left[\dot{S}_2 + \tfrac{2}{3}(S_2 d_2 - S_2 d_1 - S_1 d_2 - 2S_1 d_1)\right] \\
&\qquad = \phi_1 d_2 + \phi_2\left[\tfrac{1}{3}d_2^2 - \tfrac{2}{3}d_1^2 - \tfrac{2}{3}d_1 d_2\right] + \phi_3 \dot{d}_2 ,
\end{aligned}
\tag{23}
$$

with the third principal component just their sum.

Varying three stresses $\sigma_1(t), \sigma_2(t)$, $\sigma_3(t)$ independently gives two independent deviators $S_1(t)$, $S_2(t)$, plus an independent mean pressure $p(t) = -\frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$. However, by the incompressibility assumption, (9) is independent of p, so that variation of p provides no further information, except confirmation or rejection of the incompressibility approximation. Thus, for correlation with the model (9), it is sufficient to prescribe two independent stresses, provided that the corresponding deviators S_1, S_2, are independent.

5. Bi-axial stress

A conventional restriction to two independent stress components is $\sigma_3 \equiv \sigma_2$, discussed for a viscous fluid law by Morland (1979) and shown to provide no more information than uni-axial stress response. This conclusion will now be demonstrated for the viscoelastic fluid law (9), for all time variations $\sigma_1(t)$, $\sigma_2(t)$. However, a curve in the domain of invariants corresponding to tensile response in uni-axial stress, can be covered with $\sigma_1 < 0$, $\sigma_2 < 0$.

Now

$$S_1 = \frac{2}{3}(\sigma_1 - \sigma_2), \quad S_2 = S_3 = -\frac{1}{2}S_1 \ ,$$
$$J_2 = \frac{3}{4}S_1^2 = \frac{1}{3}(\sigma_1 - \sigma_2)^2 \ , \ J_3 = \frac{1}{4}S_1^3 = \pm\ 2(J_2/3)^{\frac{3}{2}} \ , \tag{24}$$
$$d_3 = d_2 = -\frac{1}{2}d_1 \ , \quad I_2 = \frac{3}{4}d_1^2 \ , \quad I_3 = \frac{1}{4}d_1^3 = \pm\ 2(I_2/3)^{\frac{3}{2}} \ ,$$

so that S_2, S_1, and d_2, d_1, are not independent. The two relations (23) provide only one independent relation

$$\psi_1 S_1 + \psi_3\left[\dot{S}_1 + S_1 d_1\right] = \phi_1 d_1 + \frac{1}{2}\phi_2 d_1^2 + \phi_3 \dot{d}_1 \ . \tag{25}$$

This is also the relation for uni-axial compressive stress σ where

$$\sigma_2 = \sigma_3 = 0, \quad \sigma_1 = -\sigma \ , \quad S_1 = -\frac{2}{3}\sigma \ , \quad J_2 = \frac{1}{3}\sigma^2 \ , \ J_3 = -\frac{2}{27}\sigma^3, \quad d_1 = -r \ , \tag{26}$$

which yields (12) and (13). Tension corresponds to $\sigma < 0$; that is, $J_3 > 0$ and hence the branch $J_3 = +\ 2(J_2/3)^{\frac{3}{2}}$. While uni-axial tension tests are not easy, the bi-axial test with $\sigma_2 < \sigma_1 < 0$ covers $J_3 > 0$. However, for any variation $\sigma_1(t)$, $\sigma_2(t)$, the same loading history $S_1(t)$ can be realised by appropriate $\sigma_1(t)$ in uni-axial stress, and only the same two-branch curve in the $J_2 - J_3$ plane is covered.

6. A free lateral direction, $\sigma_3 \equiv 0$.

Consider $\sigma_3 \equiv 0$, $\sigma_2 = \lambda(t)\sigma_1$, then

$$S_1 = \frac{2}{3}\sigma_1 - \frac{1}{3}\sigma_2 = \frac{1}{3}\sigma_1(2 - \lambda), \quad S_2 = \frac{2}{3}\sigma_2 - \frac{1}{3}\sigma_1 = \frac{1}{3}\sigma_1(2\lambda - 1),$$
$$J_2 = \frac{1}{3}\sigma_1^2(\lambda^2 - \lambda + 1), \quad J_3 = \frac{1}{27}\sigma_1^3\,(2\lambda^3 - 3\lambda^2 - 3\lambda + 2), \tag{27}$$

so S_1, S_2 are independent and J_2, J_3 are independent, as $\sigma_1(t)$, $\lambda(t)$ are varied independently. In the constant stress case

σ_1 = const, σ_2 = const, so λ = const, (23) reduces to two simultaneous differential equations for d_1, d_2:

$$\begin{aligned}
&\tfrac{1}{3}\sigma_1\psi_1(2-\lambda) + \tfrac{2}{3}\sigma_1\psi_3\left[(1-\lambda)d_1 - \lambda d_2\right] \\
&\qquad = \phi_1 d_1 + \tfrac{1}{3}\phi_2\left[d_1^2 - 2d_2^2 - 2d_1d_2\right] + \phi_3\dot{d}_1, \\
&\tfrac{1}{3}\sigma_1\psi_1(2\lambda-1) - \tfrac{2}{3}\sigma_1\psi_3\left[d_1 + (1-\lambda)d_2\right] \\
&\qquad = \phi_1 d_2 + \tfrac{1}{3}\phi_2\left[d_2^2 - 2d_1^2 - 2d_1d_2\right] + \phi_3\dot{d}_2 .
\end{aligned} \tag{28}$$

Given data $d_1(t)$, $d_2(t)$, and hence $\dot{d}_1$, $\dot{d}_2$ as functions of d_1, d_2, σ, λ, (28) in principle provide two relations between the coefficients ψ_1, ψ_3, ϕ_1, ϕ_2, ϕ_3, in general, functions of J_2, J_3, I_2, I_3, and rates. Normalising by $\psi_1 = 1$, if any necessary rate dependence is incorporated in ψ_3, ϕ_3, leaves two functions to determine by constant strain-rate response.

As in the uni-axial response, non-monotonic $d_1(t)$, $d_2(t)$ will impose two or more branches for $\dot{d}_1$, $\dot{d}_2$. Since $\lambda = 0$ is the uni-axial case, with its double branch for $\dot{d}_1$, we must expect such behaviour for $\dot{d}_1$, probably $\dot{d}_2$, as $|\lambda|$ increases from zero, possibly over a large range or for all λ when $\lambda > 0$; both stresses compressive or both tensile. A combination of positive and tensile stresses, $\lambda < 0$, may induce different shaped response $d_1(t)$ and $d_2(t)$. An immediate question of importance is whether data (unpublished) is currently available to indicate the shapes $d_1(t)$, $d_2(t)$ in such a loading configuration, in particular, has the shape of $d_2(t)$ been recorded in the uni-axial stress case? Once response shape is clear, the simultaneous differential equations (28) must be studied further in the spirit of the uni-axial analysis, to determine the required structure so that correlation with response is compatible. Plausible response shapes could be adopted to initiate and gain experience of such analysis, <u>if Shell wants to pursue the theoretical development of model construction at this stage</u>. Can any plausible shapes be inferred from existing data, or anticipated from other experiences?

For constant strain-rates d_1, d_2, with $d_2 = vd_1$,

$$I_2 = d_1^2(v^2 + v + 1), \quad I_3 = - vd_1^3(v + 1), \tag{29}$$

and (23) reduce to two simultaneous differential equations for S_1, S_2:

$$\psi_1 S_1 + \tfrac{2}{3}d_1\psi_3\left[(1 - v)S_1 - (1 + 2v)S_2\right] + \psi_3\dot{S}_1 = \phi_1 d_1 + \tfrac{1}{3}d_1^2\phi_2(1 - 2v - 2v^2),$$

$$(30)$$

$$\psi_1 S_2 - \tfrac{2}{3}d_1\psi_3\left[(2 + v)S_1 + (1 - v)S_2\right] + \psi_3\dot{S}_2 = \phi_1 d_2 - \tfrac{1}{3}d_1^2\phi_2(2 + 2v - v^2) \quad .$$

The lateral constraint case $d_2 \equiv 0$ $(v = 0)$ will be easier in practice than controlling two independent strain-rates, but only leaves one strain-rate invariant I_2, since $I_3 \equiv 0$ as in simple shear strain motion. Response over the full $I_2 - I_3$ domain cannot be detected. It is not clear that a combination test with $\sigma_1 = \text{const}$, $d_2 = \text{const}$ leads to tractable simultaneous equations for correlation purposes since both S_1 and S_2 still vary with t, but possibly this and further applied stress tests with varying $S_1(t)$, $S_2(t)$ will be necessary if controlling $d_1(t)$, $d_2(t)$ independently is not practical, at the expense of more difficult analysis.

7. Stress invariants domain

It remains to demonstrate that the case $\sigma_3 \equiv 0$, $\sigma_2 = \lambda\sigma_1$, with $\sigma_1 \leqslant 0$, $\sigma_2 \leqslant 0$ (both compressive) will cover the complete J_2, J_3 domain, including $J_3 > 0$, if the complete range of constant $\sigma_1 < 0$, constant $\lambda > 0$, can be applied. By construction $J_2 \geqslant 0$, and is zero only if $\sigma_1 = \sigma_2 = 0$. Define

$$\begin{aligned}
\bar{J}_2 &= \frac{3J_2}{\sigma_1^2} = \frac{3}{4} + (\lambda - \tfrac{1}{2})^2 \ , \\
\bar{J}_3 &= \frac{27J_3}{2\sigma_1^3} = \tfrac{1}{2}(1 + \lambda)(2 - \lambda)(1 - 2\lambda) = -(\lambda - \tfrac{1}{2})\left[\tfrac{9}{4} - (\lambda - \tfrac{1}{2})^2\right], \qquad (31) \\
\frac{9J_3}{2J_2} &= \frac{\bar{J}_3}{\bar{J}_2}\,\sigma_1 = k(\lambda)\sigma_1 \ , \qquad k(0) = 1 \ .
\end{aligned}$$

Now $(31)_{1,2}$ imply a relation $\bar{J}_3 = h(\bar{J}_2)$, though h is not single-valued, and hence

$$J_3 = \frac{2}{27}\,\sigma_1^3\, h\left(\frac{3J_2}{\sigma_1^2}\right) \ . \qquad (32)$$

This reduces to an explicit $J_3 - J_2$ relation, or fixed curve in the $J_2 - J_3$ plane if, and only if, $(\partial J_3/\partial\sigma_1)|_{J_2} = 0$; that is

$$\bar{J}_3 = h(\bar{J}_2) = K\,\bar{J}_2^{\frac{3}{2}} \ . \qquad (33)$$

Clearly (33) is not satisfied by the expressions $(31)_{1,2}$, and so variation of σ_1 and λ is not confined to a fixed $J_2 - J_3$ curve, but (33) is satisfied by the bi-axial stress expressions (24) with $K = \pm 1$.

$\bar{J}_2$ has a single minimum $\frac{3}{4}$ at $\lambda = \frac{1}{2}$, and is strictly positive. $\bar{J}_3$ has zeros at λ_i $(i = 1,2,3)$ and a maximum and minimum at λ_+, λ_-

respectively, given by

$$\lambda_{1,2,3} = -1,\ 0{\cdot}5,\ 2; \quad \lambda_{\pm} = \tfrac{1}{2}(1 \mp \sqrt{3}) \approx \begin{matrix}-0{\cdot}366 \\ 1{\cdot}366\end{matrix}\ , \tag{34}$$

with

$$\bar{J}_3(\lambda_{\pm}) = \pm\,\frac{3\sqrt{3}}{4} \approx \pm\ 1{\cdot}299\ . \tag{35}$$

$\bar{J}_2 \sim \lambda^2$ and $\bar{J}_3 \sim \lambda^3$ as $\lambda \to \pm\infty$, so $k(\lambda) \sim \lambda$. Figure 3 shows sketches of $\bar{J}_2(\lambda)$, $\bar{J}_3(\lambda)$, $k(\lambda)$, all symmetric about $\lambda = \frac{1}{2}$.
$k(\lambda)$ has a maximum $k_M \approx 1{\cdot}02$ at λ $-0{\cdot}1$ and a minimum $k_m \approx -1{\cdot}02$ at $\lambda \approx 1{\cdot}1$. Specifying λ determines unique $\bar{J}_2$, $\bar{J}_3$, k, then specifying σ_1 determines J_2, J_3 from (31). Alternatively, if J_2, J_3 are specified, hence $k(\lambda)\sigma_1$ and $(\lambda^2 - \lambda + 1)\sigma_1^2$, eliminating σ_1 yields a 6-tuple equation for λ with (probably) more than one real root. However, data correlation determines function values at points (σ_1,λ) for which the corresponding point (J_2,J_3) is readily calculated, and function fitting or tabulation is a direct numerical calculation in the $J_2 - J_3$ plane. It is not necessary to construct analytic functions of (σ_1,λ) and determine σ_1, λ as functions of (J_2, J_3).

If only compressive stresses $\sigma_1 < 0$, $\sigma_2 < 0$ $(\lambda > 0)$ are considered, then tension in uni-axial stress requires $J_3 > 0$ which implies $\bar{J}_3 < 0$. Figure 3 shows that at each constant σ_1, the range $0 \geqslant \bar{J}_3 \geqslant \bar{J}_3(\lambda_-)$ is covered by $\lambda > 0$, in addition to the full compression range $\bar{J}_3 > 0$. For $\lambda \geqslant 0$, all $k \geqslant k_m$ are covered, and hence all rays $J_3 \geqslant \frac{2}{9} J_2 k_m \sigma_1$ when $\sigma_1 > 0$, and all rays $J_3 \leqslant \frac{2}{9} J_2 k_m \sigma_1$ when $\sigma_1 < 0$. Thus, for compression $\sigma_1 < 0$, since $k_m < 0$, all rays $J_3 \leqslant \frac{2}{9} J_2 |k_m \sigma_1|$ are covered, shown in Fig. 4, with the two branches of the fixed curve covered by bi-axial stress. Hence, taking increasing constant stresses $\sigma_1 < 0$ increases the slope of the limit ray so that the excluded sector decreases. The maximum excluded sector is governed by the maximum compressive stress $(-\sigma_1)$ which can be applied. Note that pure uni-axial tension lies on the bi-axial curve (upper branch), and is trivially covered.

A similar strain-rate invariant analysis for I_2, I_3 can be made, essentially by the correspondence $d_1 \leftrightarrow s_1$, $d_2 \leftrightarrow s_2$ (not $d_1 \leftrightarrow \sigma_1$, $d_2 \leftrightarrow \sigma_2$).

8. Anisotropy

It is possible that anisotropy of newly formed saline ice is significant. Any such asymmetry is associated with a reference configuration and can be described only by a viscoelastic solid law. We are presently examining on the ERO contract, simple forms of differential operator laws for solids to find the structure required to describe the qualitative responses of Figs 1 and 2 in uni-axial stress. So far, however, we have considered models isotropic in the reference configuration in order to investigate anisotropy induced in subsequent configurations by loading from the reference state. It will be necessary in the Shell programme to take the newly formed ice as a reference state, and develop initially anisotropic models. This involves defining the actual asymmetry of the newly formed ice and must be tied to observation. Do we yet have a clear picture of the asymmetry?

It is suggested that salinity is an important factor in the asymmetry, which decreases as the salt concentration decreases. The asymmetry structure (directional properties) cannot be defined by a scalar salinity factor, and must be a feature of the formation process. An appropriate model may be to relate the degree of asymmetry to salinity. An example of degree is given by the ratios of different directional moduli in a linear anisotropic elastic solid, but will be more complex here. A full theory will need to describe the variation of salinity with time, either independent or dependent on the loading history, and a careful physical description of the known behaviour is required before more elaborate laws are formulated. This is an area where detailed discussions between theorist and experimental/field observers would be valuable. Has any such theory been established, or even formulated?

An exact anisotropic frame indifferent viscoelastic solid law will apply to finite deformations, but in many applications we are concerned only with small strains (prior to rupture). It is possible to introduce some simplifying approximations for small strain, examples are the identification of initial and current particle positions after eliminating rigid body motion, and forces per unit initial and unit current area in the definition of stress. However, from observation, the laws are still significantly non-linear, and truncated expansions in small strain are not appropriate.

I still expect that non-linear integral operator laws will prove more satisfactory overall than the differential operator laws, but research in this direction has still to be conducted.

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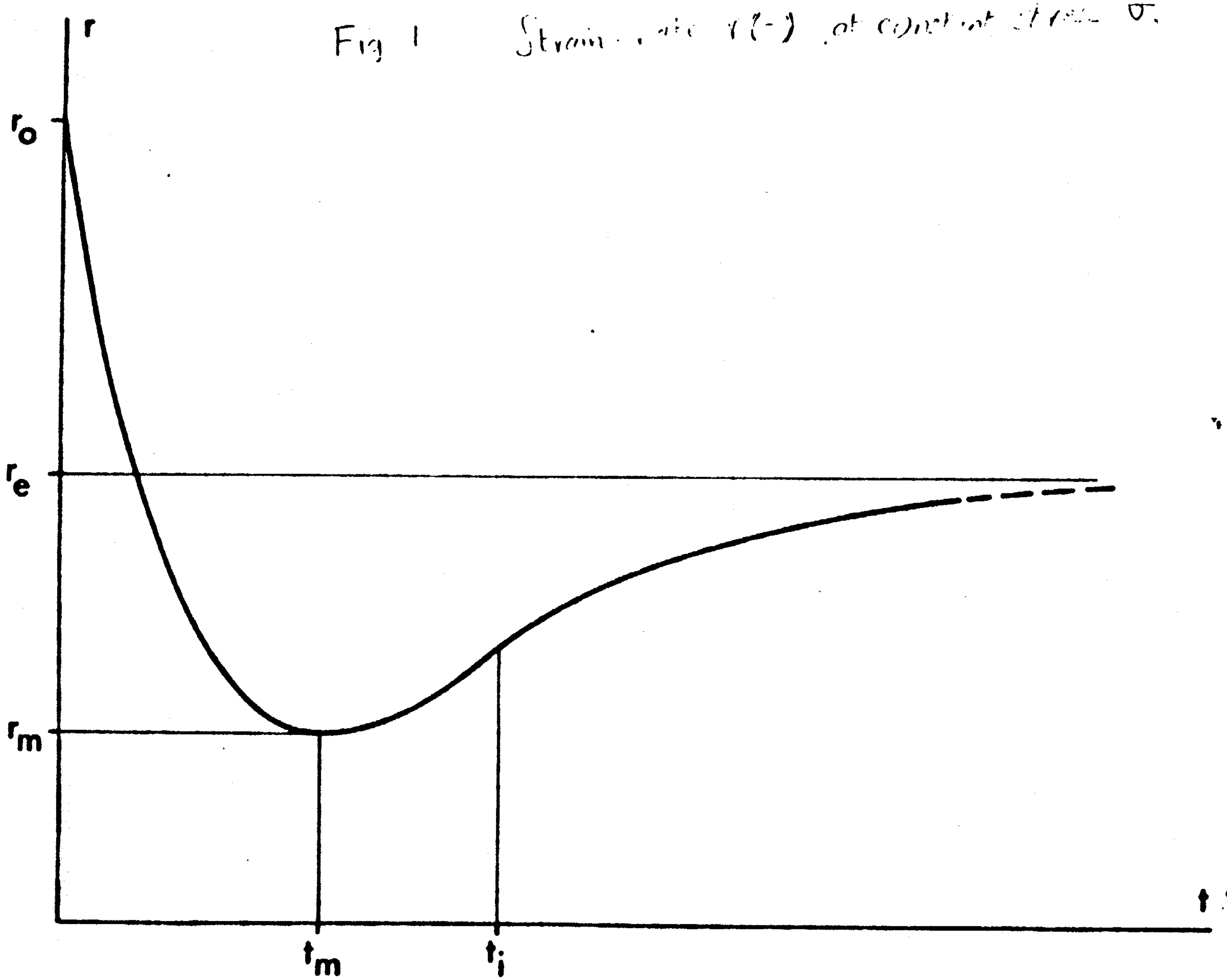
Fig 1 Strain rate r(t) at constant stress σ.



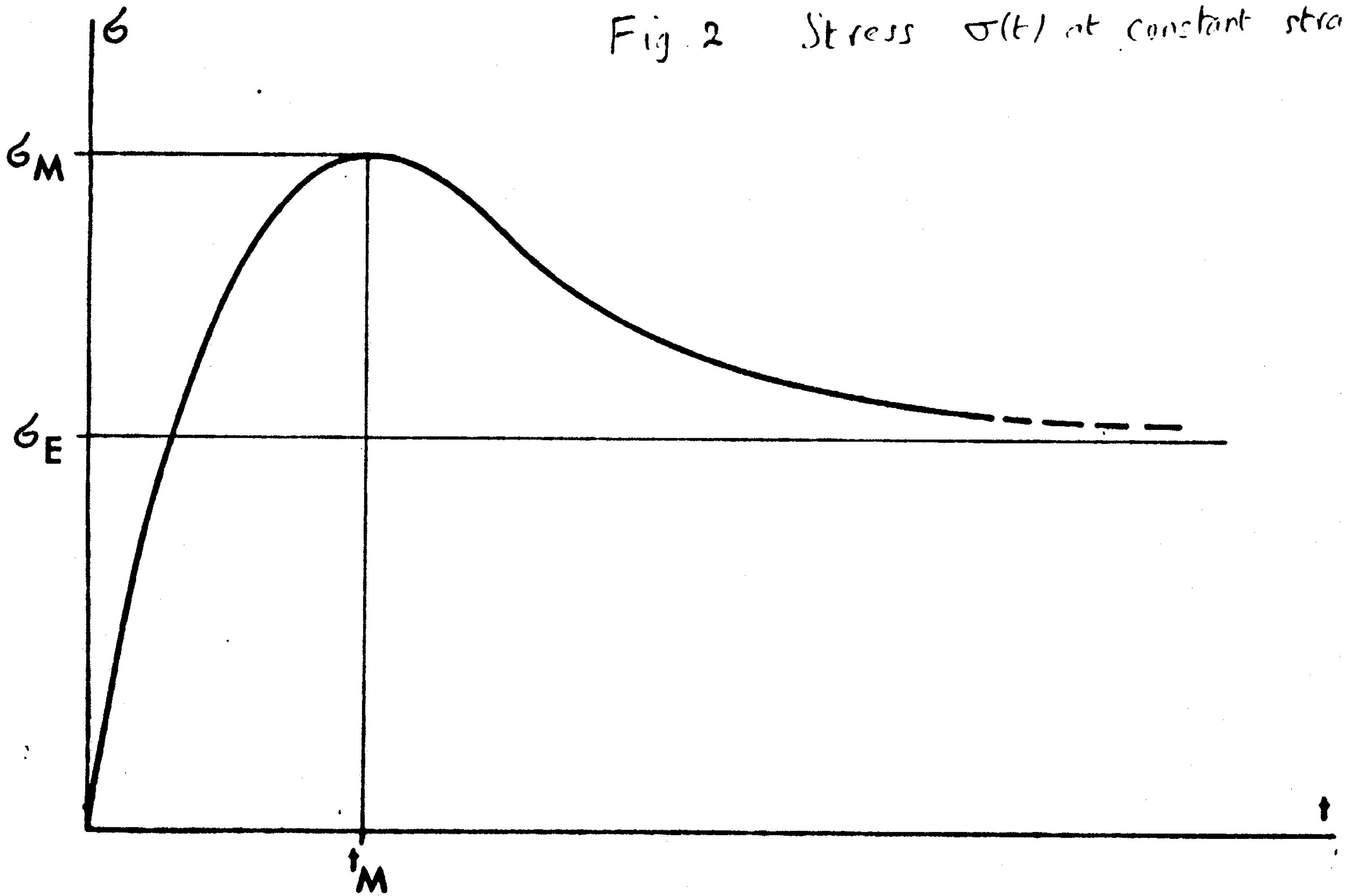


Fig. 2 Stress $\sigma(t)$ at constant strain-rate r.

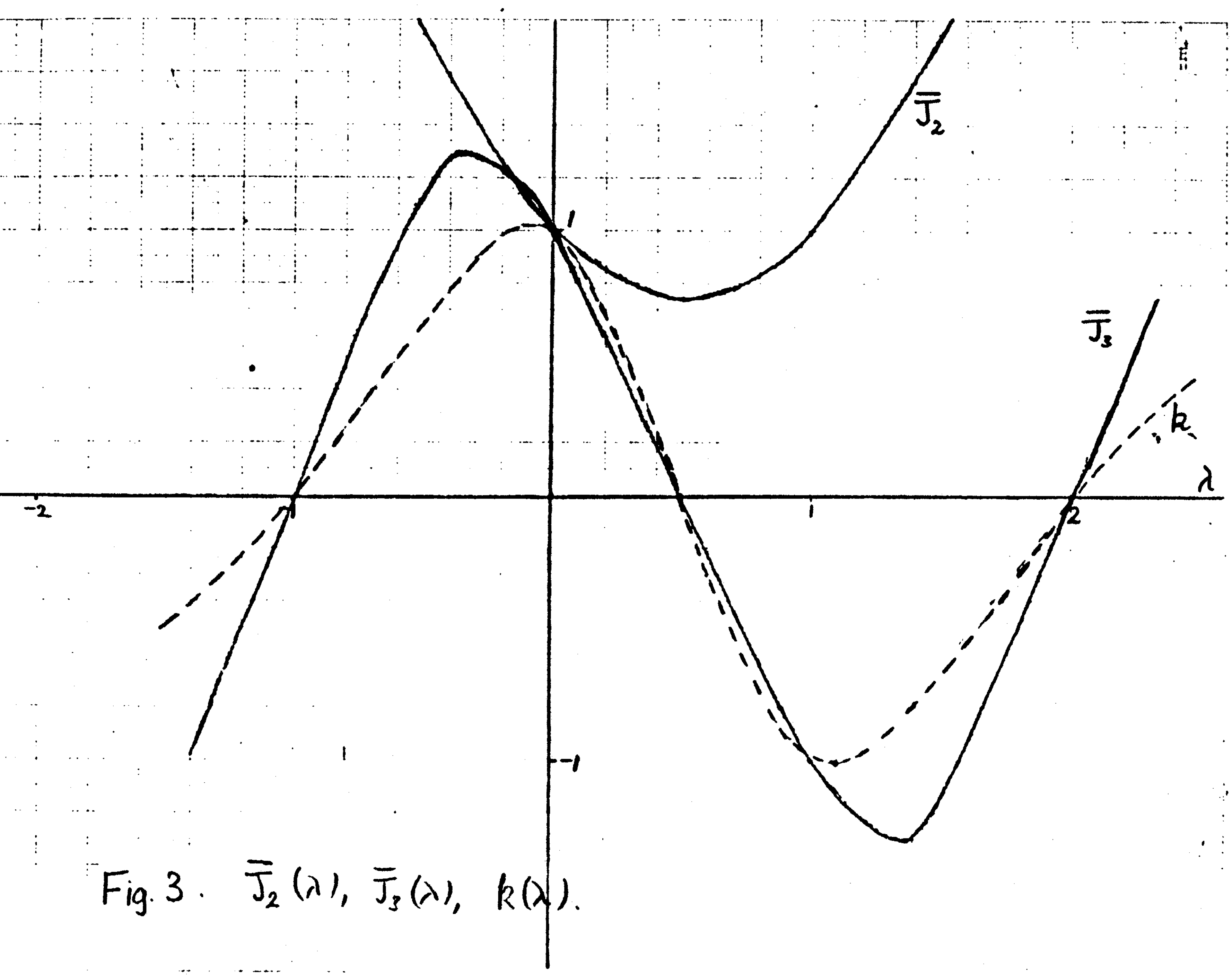


Fig. 3. $\overline{J}_2(\lambda)$, $\overline{J}_3(\lambda)$, $k(\lambda)$.

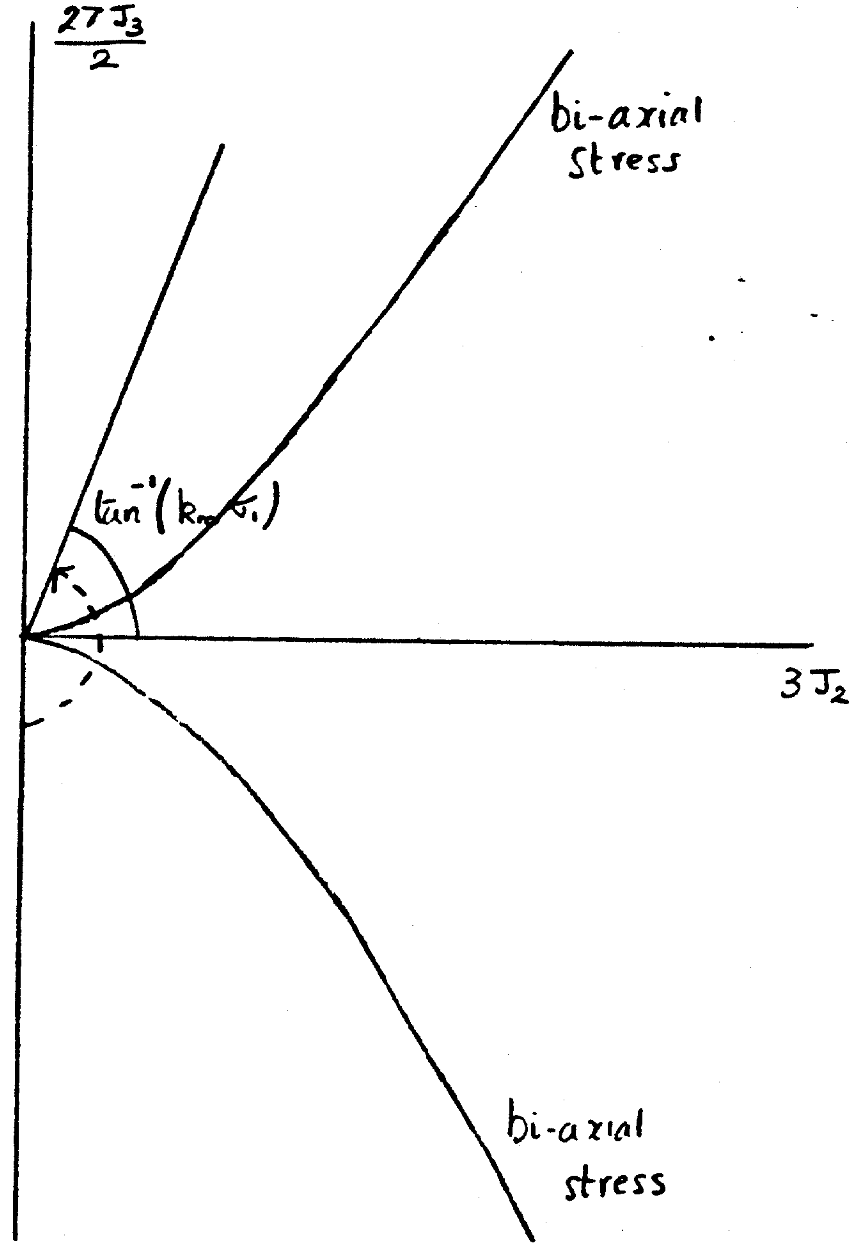


Fig. 4. Stress-invariant domains for $\sigma_1, \sigma_2 < 0$.

MECHANICAL PROPERTIES OF SEA ICE

Report M3. Compressibility and dilatancy

1. Introduction

The viscoelastic fluid model described in Report M2 assumed incompressibility. A generalisation is now presented which incorporates volume change, with illustrations for elastic compression, non-linear and linear, viscous compression, and dilatancy - volume increase under maintained shear. The response under tri-axial and bi-axial loading is analysed to investigate the extent to which such tests determine the shape and response coefficients of the model.

The generalisation adopts the differential operator law of the incompressible fluid as a relation between the stress deviator and strain-rate deviator, instead of strain-rate, so that the same viscoelastic relation is retained for the shear response. The constitutive law is completed by prescribing one of the above laws for the volume change. In the case of elastic compression, stress-acceleration terms are therebye introduced into the general relation, absent in the incompressible fluid law, but such terms are zero in constant stress and constant strain-rate response.

Response coefficients of the model can now depend on three stress invariants and three strain-rate invariants, together with their rates, so in general a complete description requires the response under three independent (tri-axial) stresses and strain-rates. In the case of dilatancy independent of pressure (induced solely by maintained shear stress), provided that the response coefficients for the shear response are also independent of pressure, a stress configuration giving two independent deviatoric stress components is sufficient. Bi-axial stress is not sufficient, but the configuration with a zero lateral stress is sufficient. However, a pressure dependent dilation can be distinguished by bi-axial stress tests provided that the shear response is pressure independent. An alternative to the zero lateral stress configuration which is more appropriate to constant strain-rate tests is the configuration with one lateral constraint, mentioned briefly in conclusion.

While the consequences of allowing volume change have been analysed only in the context of a viscoelastic fluid differential operator law, the general conclusions regarding test configurations required to describe the response must apply to other models. In the first instance the extent to which dilatancy occurs can be checked by bi-axial stress tests, and possibly the chief mechanism deduced. The complexity of more general tests may be reduced if an adequate description can be obtained by the simpler tests.

2. Volume change

If $\underset{\sim}{D} + \underset{\sim}{W}$ is the symmetric-shear decomposition of the spatial velocity gradient tensor, then the rate of increase of volume per unit volume of a material element is measured by the invariant

$$I_1 = \mathrm{tr}\,\underset{\sim}{D} = \mathrm{div}\,\underline{v} \ , = -\frac{\dot{\rho}}{\rho} \ , \tag{1}$$

where $\underline{v}$ is the particle velocity, div denotes the spatial divergence, ρ is mass density, and a superposed $\bullet$ denotes material time derivative. The Cauchy stress $\underset{\sim}{\sigma}$ can be expressed in terms of the stress deviator $\underset{\sim}{S}$, which measures shear stress, and the mean pressure p:

$$\underset{\sim}{S} = \underset{\sim}{\sigma} + p\underset{\sim}{1} , \quad p = -\frac{1}{3}\operatorname{tr}\underset{\sim}{\sigma} , \quad \operatorname{tr}\underset{\sim}{S} = 0 . \tag{2}$$

For an incompressible fluid model, $I_1 = 0$ and a constitutive relation is given for $\underset{\sim}{S}$, with p not determined by the material deformation. When volume changes occur, I_1 is a deformation variable to be related to the stress.

First consider elastic compression in which density change depends only on the mean pressure p, and not on the shear stress $\underset{\sim}{S}$. Then

$$\rho = f(p), \quad \frac{\dot{\rho}}{\rho} = \kappa(p)\dot{p} \quad \text{where} \quad \kappa(p) = \frac{f'(p)}{f(p)} . \tag{3}$$

That is,

$$I_1 = -\kappa(p)\dot{p} \tag{4}$$

where κ is the compressibility. If volume changes are very small in the pressure range of interest, then a linear approximation for $f(\rho)$, giving constant κ, is appropriate. General elastic compression allows ρ to depend also on the two shear stress invariants (a necessary restriction for frame indifference), for example

$$J_2 = \frac{1}{2}\operatorname{tr}\underset{\sim}{S}^2 , \qquad J_3 = \det\underset{\sim}{S} , \tag{5}$$

when

$$\rho = g(p,J_2,J_3), \qquad \frac{\dot{\rho}}{\rho} = \left\{\frac{\partial g}{\partial p}\dot{p} + \frac{\partial g}{\partial J_2}\dot{J}_2 + \frac{\partial g}{\partial J_3}\dot{J}_3\right\}/g. \tag{6}$$

That is,

$$I_1 = g_1(p,J_2,J_3)\dot{p} + g_2(p,J_2,J_3)\dot{J}_2 + g_3(p,J_2.J_3)\dot{J}_3, \tag{7}$$

though dependence on shear is retained with $g_3 = 0$ say. Now volume change occurs even at constant pressure, but reversibly as shear stress is applied and removed. Note that elastic relations of the form (3) or (6) imply a density jump when a stress jump is applied, though so further density change if the stress is maintained constant. Thus, applying (6), if the stress is increased from zero to $\underset{\sim}{\sigma}$ at $t = 0$ and then held constant,

$$\rho(t) \equiv g(p,J_2,J_3) \quad \text{for} \quad t > 0, \tag{8}$$

where the initial density is $\rho_o = g(0,0,0)$.

A better model of dilatancy, the opening of pores and cracks under maintained shear, including constant stress, is given by

$$I_1 = h(J_2,J_3), \quad h \geqslant 0, \tag{9}$$

or more simply a dependence on J_2 say. Such a relation determines a constant rate of volume increase per unit volume at constant shear stress, the rate depending on the stress magnitude. Similarly, if the ice has a bulk viscosity, not necessarily constant, then

$$I_1 = -\ell(p), \qquad \ell \geqslant 0, \tag{10}$$

now giving a constant rate of volume change (decrease) at constant pressure. However, in both situations there must be bounds on the maximum and minimum densities, ρ_M and ρ_m, independent of the load duration, possibly depending on the stress level. Suppose $\rho_M/\rho_o = B > 1$, $\rho_m/\rho_o = b < 1$, then

$$I_1 = q(\rho/\rho_o)h(J_3,J_3) \quad \text{or} \quad -q(\rho/\rho_o)\ell(p), \tag{11}$$

or some combination, with

$$q(x) \equiv \left\{ \begin{array}{l} 0 \quad \text{for} \quad x \leqslant b \quad \text{and} \quad x \geqslant B \\ > 0 \quad \text{for} \quad b < x < B \end{array} \right\}, \tag{12}$$

controls the permitted density range, and the rate level as the limits are approached. Now the relations (11) are implicit equations for I_1 or ρ/ρ_o, since $I_1\rho/\rho_o = -(\rho/\rho_o)^{\bullet}$.

3. Modified viscoelastic fluid model

The incompressible fluid relation (M2.9) relating stress, stress-rate, strain-rate, and strain-acceleration, with linear dependence on stress-rate and strain-acceleration tensors, is

$$\begin{aligned} &\psi_1\underset{\sim}{S} + \psi_3[\dot{\underset{\sim}{S}} + \underset{\sim}{S}(\underset{\sim}{D} + \underset{\sim}{W}) + (\underset{\sim}{D} - \underset{\sim}{W})\underset{\sim}{S} - \tfrac{2}{3}\,\mathrm{tr}\,(\underset{\sim}{S}\underset{\sim}{D})\underset{\sim}{1}] \\ &\quad = \phi_1\underset{\sim}{D} + \phi_2[\underset{\sim}{D}^2 - \tfrac{2}{3}I_2\,\underset{\sim}{1}] + \phi_3[\dot{\underset{\sim}{D}} + \underset{\sim}{D}\underset{\sim}{W} - \underset{\sim}{W}\underset{\sim}{D}], \end{aligned} \tag{13}$$

where the coefficients are, in general, functions of the invariants

$$\begin{gathered} J_2 = \tfrac{1}{2}\,\mathrm{tr}\,\underset{\sim}{S}^2, \quad J_3 = \det \underset{\sim}{S}, \quad I_2 = \tfrac{1}{2}\,\mathrm{tr}\,\underset{\sim}{D}^2, \quad I_3 = \det \underset{\sim}{D}, \\ \dot{J}_2, \ \dot{J}_3, \ \dot{I}_2, \ \dot{I}_3. \end{gathered} \tag{14}$$

We now have additional invariants I_1 and p, and $\dot{I}_1$, $\dot{p}$, which may enter the coefficient dependence, and an additional relation for I_1. In the dilatancy model (9) or $(11)_1$ there is no explicit I_1 dependence on p.

Note that each tensor term multiplying a response coefficient in the relation (13) between a deviatoric stress $\underset{\sim}{S}$ $(\mathrm{tr}\,\underset{\sim}{S} = 0)$ and a deviatoric strain-rate $\underset{\sim}{D}$ $(\mathrm{tr}\,\underset{\sim}{D} = 0$ for incompressible fluid) was constructed to have zero trace. It is convenient to retain this form of relation between the stress deviator $\underset{\sim}{S}$ and the frame indifferent strain-rate deviator

$$\bar{\underset{\sim}{D}} = \underset{\sim}{D} - \frac{1}{3} I_1 \underset{\sim}{1} \ , \qquad \mathrm{tr}\,\bar{\underset{\sim}{D}} = 0 \ , \tag{15}$$

to reflect the viscoelastic fluid nature of the shear response and add the relation for the dilatation rate I_1 to complete the determination of $\underset{\sim}{D}$. Thus

$$\begin{aligned} \bar{\psi}_1 \underset{\sim}{S} + \bar{\psi}_3 \left[\dot{\underset{\sim}{S}} + \underset{\sim}{S}(\bar{\underset{\sim}{D}} + \underset{\sim}{W}) + (\bar{\underset{\sim}{D}} - \underset{\sim}{W})\underset{\sim}{S} - \frac{2}{3}\mathrm{tr}\,(\underset{\sim}{S}\bar{\underset{\sim}{D}})\underset{\sim}{1} \right] \\ = \bar{\phi}_1 \bar{D} + \bar{\phi}_2 \left[\bar{D}^2 - \frac{2}{3} \bar{I}_2 \underset{\sim}{1} \right] + \bar{\phi}_3 \left[\dot{\bar{D}} + \bar{\underset{\sim}{D}}\underset{\sim}{W} - \underset{\sim}{W}\bar{\underset{\sim}{D}} \right] , \end{aligned} \tag{16}$$

where

$$\bar{I}_2 = \frac{1}{2} \mathrm{tr}\,\bar{\underset{\sim}{D}}^2 = I_2 - \frac{1}{6} I_1^2 \ , \quad \bar{I}_3 = \det \bar{\underset{\sim}{D}} \ , \tag{17}$$

and the new response coefficients $\bar{\psi}_1$ etc are functions of the invariants J_2, J_3, $\bar{I}_2$, $\bar{I}_3$, and p and I_1, together with their rates. The relation (16) can be expressed directly in terms of $\underset{\sim}{D}$, and invariants I_2, I_3 used instead of $\bar{I}_2$, $\bar{I}_3$.

4. Tri-axial stress

For three independent principal stresses $\sigma_1, \sigma_2, \sigma_3$, and three independent principal strain-rates d_1, d_2, d_3 :

$$
\begin{aligned}
&p = -\tfrac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3),\ S_1 = \sigma_1 + p,\ S_2 = \sigma_2 + p,\ S_3 = -(S_1 + S_2),\\
&I_1 = d_1 + d_2 + d_3,\ \bar{d}_1 = d_1 - \tfrac{1}{3}I_1,\ \bar{d}_2 = d_2 - \tfrac{1}{3}I_1,\ \bar{d}_3 = -(\bar{d}_1 + \bar{d}_2),\\
&J_2 = S_1^2 + S_2^2 + S_1 S_2,\quad J_3 = -S_1 S_2 (S_1 + S_2),\\
&\bar{I}_2 = \bar{d}_1^2 + \bar{d}_2^2 + \bar{d}_1\bar{d}_2,\quad \bar{I}_3 = -\bar{d}_1\bar{d}_2(\bar{d}_1 + \bar{d}_2)\ .
\end{aligned}
\tag{18}
$$

Hence, with zero rotation, (16) yields two independent relations

$$
\begin{aligned}
&\bar{\Psi}_1 S_1 + \bar{\Psi}_3\left[\dot{S}_1 + \tfrac{2}{3}(S_1\bar{d}_1 - S_2\bar{d}_1 - S_1\bar{d}_2 - 2S_2\bar{d}_2)\right]\\
&\qquad = \bar{\Phi}_1 d_1 + \bar{\Phi}_2\left[\tfrac{1}{3}\bar{d}_1^2 - \tfrac{2}{3}\bar{d}_2^2 - \tfrac{2}{3}\bar{d}_1\bar{d}_2\right] + \bar{\Phi}_3\dot{\bar{d}}_1\ ,\\
&\bar{\Psi}_1 S_2 + \bar{\Psi}_3\left[\dot{S}_2 + \tfrac{2}{3}(S_2\bar{d}_2 - S_2\bar{d}_1 - S_1\bar{d}_2 - 2S_1\bar{d}_1)\right]\\
&\qquad = \bar{\Phi}_1\bar{d}_2 + \bar{\Phi}_2\left[\tfrac{1}{3}\bar{d}_2^2 - \tfrac{2}{3}\bar{d}_1^2 - \tfrac{2}{3}\bar{d}_1\bar{d}_2\right] + \bar{\Phi}_3\dot{\bar{d}}_2\ ,
\end{aligned}
\tag{19}
$$

with the third relation just the sum of the relations (19) because each tensor combination in (16) has zero trace. These relations are analogus to the S_1, S_2, d_1, d_2 relations (M2.23), but now the response coefficient can depend also on the variable invariants I_1 and p. Also $\bar{d}_1, \bar{d}_2$ depend on I_1 as well as d_1, d_2; that is, on the three strain-rates d_1, d_2, d_3.

Measurement of three stresses and three strain-rates is required to describe the two deviatoric relations (19) together with a volume change relation. In the incompressible case, $I_1 = 0$, p arbitrary, the two relations (M2.23) require only S_1, S_2, and d_1, d_2, to be independent in order to describe the response, which was possible with two independent stresses. If the relation for I_1 is already known, however, then measurement of the response for independent S_1 and S_2 is sufficient to determine the deviatoric response (16). Since $\dot{I}_1$ occurs in (19), through $\dot{\bar{d}}_1, \dot{\bar{d}}_2$, the elastic compressibility laws (4) and (16) introduce dependence on $\ddot{p}$ or $\ddot{J}_2$, $\ddot{J}_3$; that is, on stress-acceleration, which are higher derivatives than arise in the incompressible model. However, in constant stress tests, $\ddot{p} = \ddot{J}_2 = \ddot{J}_3 = 0$, and in constant strain-rate tests $\dot{I}_1 = 0$, so these higher derivative terms do not arise in the differential relations describing the response.

It is worth noting that the elastic and viscous volume change models suggested in section 2 exhibit no time variation of I_1 at any given constant stress, in contrast to the primary-secondary-tertiary creep phases of the expected deviatoric response. Such time-independent strain-rate would be distinguishable from the viscoelastic response, and so a dilatation model may emerge even when only two independent stresses are varied.

5. Bi-axial stress

Conventional bi-axial stress implies $\sigma_3 \equiv \sigma_2$, $d_3 \equiv d_2$,

$$p = -\frac{1}{3}(\sigma_1 + 2\sigma_2), \quad S_1 = \frac{2}{3}(\sigma_1 - \sigma_2), \; S_2 = S_3 = -\frac{1}{2}S_1,$$

$$I_1 = d_1 + 2d_2, \; \bar{d}_1 = \frac{2}{3}(d_1 - d_2), \; \bar{d}_2 = \bar{d}_3 = -\frac{1}{2}\bar{d}_1 \,, \tag{20}$$

$$J_2 = \frac{1}{3}(\sigma_1 - \sigma_2)^2, \; J_3 = \frac{2}{27}(\sigma_1 - \sigma_2)^3, \; \bar{I}_2 = \frac{1}{3}(d_1 - d_2)^2,$$

$$\bar{I}_3 = \frac{2}{27}(d_1 - d_2)^3.$$

Now S_2, S_1, and $\bar{d}_2, \bar{d}_1$ are not independent, and the two relations (19) provide only one independent relation

$$\bar{\psi}_1 S_1 + \bar{\psi}_3\left[\dot{S}_1 + S_1\bar{d}_1\right] = \bar{\phi}_1\bar{d}_1 + \frac{1}{2}\bar{\phi}_2\bar{d}_1^2 + \bar{\phi}_3\dot{\bar{d}}_1 \,, \tag{21}$$

analogous to (M2.25) but with response coefficient dependence on I_1 and p also. The identity of the second and third components, and zero trace of each term of (16), requires that the second component is half the negative of the first component for each term.

Thus only one relation between S_1 and $\bar{d}_1$ is given, which can describe dependence on one deviatoric stress invariant and one deviatoric strain-rate invariant only. However, measurement of the two stresses σ_1, σ_2 and two strain-rates d_1, d_2, also determines I_1 and p, and will determine any bulk viscosity relation between I_1 and p or any dilatancy relation between I_1 and J_2 or I_1 and J_3, or some combination of J_2, J_3. <u>Thus, a bi-axial stress test can distinguish a dilatation response from a deviatoric response</u>. In a uni-axial stress test, $\sigma_2 = \sigma_3 = 0$, $S_1 = \frac{2}{3}\sigma_1$, $p = -\frac{1}{3}\sigma_1 = -\frac{1}{2}S_1$, but I_1 and $\bar{d}_1$ are independent, so while dilatation and shear strain rates can be distinguished, dependence on p or S_1 cannot be distinguished.

6. A free lateral direction, $\sigma_3 \equiv 0$.

For $\sigma_3 \equiv 0$ and $\sigma_2 = \lambda(t)\sigma_1$,

$$S_1 = \frac{1}{3}\sigma_1(2-\lambda),\ S_2 = \frac{1}{3}\sigma_1(2\lambda-1),\ p = -\frac{1}{3}\sigma_1(1+\lambda),$$
$$J_2 = \frac{1}{3}\sigma_1^2(\lambda^2-\lambda+1),\ J_3 = \frac{1}{27}\sigma_1^3(2\lambda^3-3\lambda^2+2), \tag{22}$$

and S_1, S_2 are independent as σ_1, λ are varied independently. The analysis (M2.§7) of the stress invariants domain shows that a full J_2, J_3 domain, including tensile conditions, can be covered by compressive principal stresses, $\sigma_1 \leqslant 0$, $\sigma_2 \leqslant 0$. But now, in contrast to bi-axial stress, while J_2, J_3 are independent, p is a function of J_1 and J_2 so that dependence on all three stress invariants J_2, J_3, p, cannot be distinguished. That is, varying σ_1 and σ_2 allows a response description only on a surface in J_2, J_3, p space. The corresponding domain for bi-axial stress with $\sigma_1 \leqslant 0$, $\sigma_2 \leqslant 0$, is the surface $J_3 = \pm 2(J_{2/3})^{3/2}$, $p \geqslant 0$. If shear dependence can be restricted to J_2, then a domain $J_2 \geqslant 0$, $p \geqslant 0$ is covered by $\sigma_1 \leqslant 0$, $\sigma_2 \leqslant 0$. In the dilatancy models (9) or (11), for which there is no dependence on p, and if the deviatoric response coefficients are also assumed independent of p, then the full

J_2, J_3 domain of dependence is covered.

Constant stress tests, $\dot{\sigma}_1 = 0$, $\dot{\lambda} = 0$, again give the two differential equations (M2.28) for $\bar{d}_1, \bar{d}_2$ in place of d_1, d_2, which in turn yield two differential equations for d_1, d_2 which depend also on $d_3(t)$, $\dot{d}_3(t)$. Thus, by measuring $d_1(t)$, $d_2(t)$, and $d_3(t)$, two relations on the response coefficients are obtained as before, and with the assumption that the deviatoric response coefficients are independent of p, then the full J_2, J_3 domain of dependence is covered. The three independent d_1, d_2, d_3 determine three invariants $I_1, \bar{I}_2, \bar{I}_3$ as required.

Hence, with this assumption for the deviatoric response coefficients, constant bi-axial stress tests can be used to determine any dilatation dependence on p, then the above configuration used to determine deviatoric response.

Constant strain-rate tests in this configuration require all three rates d_1, d_2, d_3 to be controlled, which may not be practical in general. Two independent constant strain-rate tests are required to complement (M2.28) to determine the deviatoric relation. An alternative configuration is the following lateral constraint test.

7. Lateral constraint, $d_3 \equiv 0$.

For $d_3 \equiv 0$, and d_1, d_2 independent constants, there are three principal stresses $\sigma_1, \sigma_2, \sigma_3$. Provided $d_1 < 0$, $d_2 < 0$, representing compression in both directions, $d_3 \equiv 0$ can be maintained by a rigid constraint supplying the necessary compressive stress $\sigma_3 < 0$. From (18) and (19),

$$I_1 = d_1 + d_2,\ \bar{d}_1 = \tfrac{2}{3}d_1 - \tfrac{1}{3}d_2,\ \bar{d}_2 = \tfrac{2}{3}d_2 - \tfrac{1}{3}d_1,$$
$$\bar{I}_2 = \tfrac{1}{3}(d_1^2 - d_1d_2 + d_2^2),\ \bar{I}_3 = -\tfrac{1}{27}(d_1 + d_2)(2d_1 - d_2)(2d_2 - d_1), \tag{23}$$

and the two differential equation for S_1, S_2 are

$$\bar{\psi}_1 S_1 + \tfrac{2}{3}\bar{\psi}_3 S_1(d_1 - d_2) - \tfrac{2}{3}\bar{\psi}_3 S_2 d_2 + \bar{\psi}_3 \dot{S}_1$$
$$= \tfrac{1}{3}\bar{\phi}_1\,(2d_1 - d_2) + \tfrac{1}{9}\bar{\phi}_2\,(2d_1^2 - 2d_1d_2 - d_2^2) \tag{24}$$
$$\bar{\psi}_1 S_2 - \tfrac{2}{3}\bar{\psi}_3 S_2(d_1 - d_2) - \tfrac{2}{3}\bar{\psi}_3 S_1 d_1 + \bar{\psi}_3 \dot{S}_2$$
$$= \tfrac{1}{3}\bar{\phi}_1\,(2d_2 - d_1) + \tfrac{1}{9}\bar{\phi}_2\,(2d_2^2 - 2d_1d_2 - d_1^2).$$

MECHANICAL PROPERTIES OF SEA ICE

Report M5 Pressure Dependence

1. Introduction

Report M2 described a viscoelastic fluid differential operator law to describe the mechanical response of ice under the assumption of incompressibility. In this case, the mean pressure is not determined by the deformation and the law determines only the deviatoric response. Various compressibility/dilatancy models were discussed in Report M3 which relate volume change to pressure or shear stress, or both, and these can be appended to the deviatoric relation. However, the simplifying incompressibility assumption may be a good approximation in many practical situations, but increasing the mean pressure can increase the ductility by inhibiting microcrack development. For example, the constant strain-rate response in uniaxial compressive stress, illustrated by Figure 2 in M2, may exhibit an increased peak stress σ_M and the smooth increase to σ_M actually shown there, if the test is conducted under a superposed isotropic pressure. That is, the deviatoric (shear) response is influenced by mean pressure.

Thus, in a fluid model (necessarily isotropic), the response coefficients will depend on p in addition to two shear stress invariants, J_2, J_3 (M2.11), as well as the two strain-rate invariants I_2, I_3. In a viscoelastic solid model (Spring and Morland, 1981) with respect to an isotropic reference configuration, the response coefficients then depend on p, J_2, J_3, as well as two strain invariants K_1, K_2 (S&M.6). <u>The dependence of response coefficients on three stress invariants p, J_2, J_3 can only be distinguished if test data determine the response to three independently varied stress components</u>. If shear tests are not practical, then general triaxial tests, with three independent principal stresses σ_1, σ_2, σ_3 must be conducted to confirm such dependence. Note that the principal stress configuration $(\sigma_1, \sigma_2, 0)$, arbitrary σ_1, σ_2, discussed in M2, with superposed mean pressure $(-p, -p, -p)$, arbitrary p, is equivalent to a general triaxial stress configuration $(\sigma_1, \sigma_2, \sigma_3)$, arbitrary σ_1, σ_2, σ_3.

If data only for two independent stress component tests can be obtained, an alternative interpretation to the (J_2,J_3) dependence described in Report M2 is the interpretation as (J_2,p) dependence. That is, we consider a model in which deviatoric response coefficients depend only on one deviatonic stress invariant and on the mean pressure p. Both strain rate invariants I_2, I_3 in the fluid model or both strain invariants $K_1 K_2$ in the solid model can be retained since a third deformation invariant is constant by the incompressibility assumption. The (J_2, p) domains covered by the test configurations discussed in Report M2 will be compared with the analogous (J_2, J_3) domains after clarifying the stress configuration terminology.

2. Stress Configuration Terminology

In general triaxial stress there are three independent principal stresses $(\sigma_1, \sigma_2, \sigma_3)$, so that the Cauchy stress $\underset{\sim}{\sigma}$ and deviatoric (shear) stress $\underset{\sim}{S}$ are given by

$$\underset{\sim}{\sigma} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}, \qquad \underset{\sim}{S} = \begin{pmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & (-S_1 + S_2) \end{pmatrix}, \tag{1}$$

$$S_1 = \frac{2}{3}\sigma_1 \;\; \frac{1}{3}(\sigma_2 + \sigma_3), \quad S_2 = \frac{2}{3}\sigma_2 \;\; \frac{1}{3}(\sigma_1 + \sigma_3), \quad S_3 = \frac{2}{3}\sigma_3 \;\; \frac{1}{3}(\sigma_1 + \sigma_2) \quad , \tag{2}$$

with mean pressure p and deviatoric invariants given by

$$\begin{aligned} p &= -\frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) , \\ J_2 &= \frac{1}{2}\,\mathrm{tr}\,\underset{\sim}{S}^2 = \frac{1}{3}[\sigma_1{}^2 + \sigma_2{}^2 + \sigma_3{}^2 \;\; (\sigma_2\sigma_3 + \sigma_3\sigma_1 + \sigma_1\sigma_2)] , \\ J_3 &= \det S = \frac{1}{27}[(2\sigma_1 - \sigma_2 - \sigma_3)(2\sigma_2 - \sigma_1 - \sigma_3)(2\sigma_3 - \sigma_1 - \sigma_2)] . \end{aligned} \tag{3}$$

By convention, principal stresses are positive in tension. The general triaxial configuration defined by (1) - (3) will be described as <u>triaxial stress, abbreviated to TS</u>.

A conventional triaxial stress configuration usually refers to the restricted case of two independent stresses σ_1, σ_2 with $\sigma_3 \equiv \sigma_2$. This was designated biaxial stress by Morland (1980), but this terminology is better reserved for stress applied along two axes only. Let us introduce the precise description transversely isotropic stress, abbreviated to TIS, since the stress in planes transverse to the longitudinal principal stress σ_1 axis is isotropic. For TIS,

$$\underset{\sim}{\sigma} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_2 \end{pmatrix}, \quad \underset{\sim}{S} = \frac{1}{3}\begin{pmatrix} 2(\sigma_1-\sigma_2) & 0 & 0 \\ 0 & \sigma_2-\sigma_1 & 0 \\ 0 & 0 & \sigma_2-\sigma_1 \end{pmatrix}, \tag{4}$$

$$p = -\tfrac{1}{3}(\sigma_1 + 2\sigma_2), \qquad J_2 = \tfrac{1}{3}(\sigma_1 - \sigma_2)^2, \qquad J_3 = \tfrac{2}{27}(\sigma_1 - \sigma_2)^3. \tag{5}$$

Thus, J_2 and J_3 are not independent, but

$$J_3 = \pm 2(J_2/3)^{3/2} \quad . \tag{6}$$

Uniaxial stress, abbreviated to US, is given by $\sigma_2 = \sigma_3 = 0$:

$$p = -\tfrac{1}{3}\sigma_1, \quad S_1 = \tfrac{2}{3}\sigma_1, \quad S_2 = S_3 = -\tfrac{1}{3}\sigma_1, \quad J_2 = \tfrac{1}{3}\sigma_1^2, \quad J_3 = \tfrac{2}{27}\sigma_1^3, \tag{7}$$

so that in uniaxial compression ($\sigma_1 < 0$) $J_3 < 0$ and in uniaxial tension ($\sigma_1 > 0$) $J_3 > 0$.

Finally, biaxial stress, abbreviated to BS, will refer to the case $\sigma_3 = 0$; that is, one lateral direction is free and stresses σ_1 and σ_2 are applied longitudinally and in one lateral direction. Now

$$\underset{\sim}{\sigma} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \underset{\sim}{S} = \frac{1}{3}\begin{pmatrix} 2\sigma_1-\sigma_2 & 0 & 0 \\ 0 & 2\sigma_2-\sigma_1 & 0 \\ 0 & 0 & -(\sigma_1+\sigma_2) \end{pmatrix}, \tag{8}$$

$$p = -\tfrac{1}{3}(\sigma_1 + \sigma_2), \quad J_2 = \tfrac{1}{3}(\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2),$$

$$J_3 = \tfrac{1}{27}(2\sigma_1^3 - 3\sigma_1^2\sigma_2 - 3\sigma_1\sigma_2^2 + 2\sigma_2^3) . \tag{9}$$

Here J_2 and J_3 are independent. Uniaxial stress (7) is recovered by setting $\sigma_2 = 0$.

3. Stress Invariant Domains in TIS and BS

In both TIS and BS, there are two independent stresses, σ_1 and σ_2. It is convenient to set

$$\sigma_2(t) = \lambda(t)\sigma_1(t) \tag{10}$$

and regard σ_1,λ as the independent parameters. We can examine the more practical situation $\sigma_1 \leq 0$, $\sigma_2 \leq 0$ ($\lambda \geq 0$) when both applied stresses are compressive, to determine the (J_2,p) domain covered by such tests. $J_2 \geq 0$ by definition and with the restriction $\sigma_1 \leq 0$, $\sigma_2 \leq 0$, the mean pressure $p \geq 0$ for TIS (5) and BS (9). Thus, pure uniaxial tension,

$$US(\sigma_1 > 0): \quad p = -(J_2/3)^{1/2} < 0 \quad , \tag{11}$$

is not covered by compressive tests, in contrast to

$$US(\sigma_1 > 0): \quad J_3 = 2(J_2/3)^{3/2} > 0 \quad , \tag{12}$$

covered in both TIS and BS tests (Report M2, Figure 4). In the (J_2,J_3) domain, TIS (described as biaxial stress in Figure 4 of M2) lies on the two branches of the curve (b), while BS covers a sector

$$J_3 \leq \frac{2}{9} J_2 \, |k_m\sigma_1| \, , \qquad \text{constant } k_m \quad , \tag{13}$$

which allows increasing positive J_3 as $(-\sigma_1)$ increases. Hence, interpretation of TIS and BS compression data as (J_2, p) dependence does not describe the response in uniaxial tension.

First, consider TIS and express (5) in the notation (10):

$$p = -\frac{1}{3}\sigma_1(1 + 2\lambda), \qquad J_2 = \frac{1}{3}\sigma_1{}^2(1 - \lambda)^2, \qquad J_3 = \frac{2}{27}\sigma_1{}^3(1 - \lambda)^3. \tag{14}$$

Then,

$$J_2 = \mu p^2, \qquad \mu = 3\left(\frac{1 - \lambda}{1 + 2\lambda}\right)^2 \geq 0 \quad , \tag{15}$$

and for each fixed λ, fixing μ, a parabola $(15)_1$ in the $(J_2,\, p \geq 0)$ domain is traversed as $(-\sigma_1)$ increases from zero. Now

$$\frac{d\mu}{d\lambda} = -\frac{18(1-\lambda)}{(1+2\lambda)^3} \begin{matrix} < \\ > \end{matrix} 0 \text{ as } \begin{matrix} 0 \leq \lambda < 1 \\ \lambda > 1 \end{matrix} , \tag{16}$$

and

$$\mu(\lambda = 0) = 3, \quad \mu_{min}(\lambda = 1) = 0, \quad \mu(\lambda \to \infty) = 3/4 \quad . \tag{17}$$

The range $0 \leq \mu \leq 3$ is covered by $1 \geq \lambda \geq 0$; that is, $-\sigma_2 \leq -\sigma_1$, and an interval $0 < \mu < 3/4$ is duplicated by $\lambda > 1$ when $-\sigma_2 > -\sigma_1$, providing some check on the assumption of (J_2, p) dependence only. Figure 1 illustrates the domain covered, bounded by the positive p-axis ($J_2 = 0$) and limit parabola $J_2 = 3p^2$ (uniaxial compression). Here, TIS covers a domain of the (J_2, p) plane, not just a curve.

In BS,

$$p = -\frac{1}{3}\sigma_1(1+\lambda), \quad J_2 = \frac{1}{3}\sigma_1^2(1-\lambda+\lambda^2), \quad J_3 = \frac{1}{27}\sigma_1^3(2-3\lambda-3\lambda^2+2\lambda^3) \ , \tag{18}$$

whence

$$J_2 = \nu p^2 , \quad \nu = \frac{3(1-\lambda+\lambda^2)}{(1+\lambda)^2} , \tag{19}$$

$$\frac{d\nu}{d\lambda} = -\frac{9(1-\lambda)}{(1+\lambda)^3} \begin{matrix} < \\ > \end{matrix} 0 \text{ as } \begin{matrix} 0 \leq \lambda < 1 \\ \lambda > 1 \end{matrix} , \tag{20}$$

$$\nu(\lambda = 0) = 3, \quad \nu_{min}(\lambda = 1) = 3/4, \quad \nu(\lambda \to \infty) = 3 \quad . \tag{21}$$

Thus, a more restricted range $3/4 \leq \nu \leq 3$ than for TIS is obtained, with complete duplication between $\lambda < 1$ and $\lambda > 1$ (simply the roles of σ_1 and σ_2 reversed since $\sigma_3 = 0$). The limit parabolas are shown in Figure 1, $J_2 = 3p^2$ common to both TIS and BS. Recall (M2) that TIS gives only one independent deviatoric relation for the minimal differential operator tensor law describing a viscoelastic fluid, while BS gives two independent relations necessary without further ad hoc assumptions to restrict the model. Both excluded domains in BS, $J_2 > 3p^2$ and $J_2 < (3/4)p^2$, are of practical significance, so additional test geometries are required to complement BS if complete (J_2, p) dependence is required from compression tests.

If tension can be applied in the axial direction:

$$\sigma_1 \geq 0, \qquad \sigma_2 \leq 0, \qquad \lambda \leq 0 \quad , \tag{22}$$

with compression in the lateral directions (TIS) or one lateral direction (BS), the (J_2,p) domain is extended. With (22), TIS gives

$$p \begin{smallmatrix} \leq \\ \geq \end{smallmatrix} 0 \text{ as } \begin{matrix} 0 \geq \lambda \geq -1/2 \\ \lambda \leq -1/2 \end{matrix}, \quad \frac{d\mu}{d\lambda} < 0 \text{ for } \lambda \leq 0 \ , \tag{23}$$

and

$$\mu\ (\lambda = 0) = 3, \qquad \mu(\lambda \to -1/2) \to \infty, \qquad \mu(\lambda \to -\infty) = 3/4 \quad . \tag{24}$$

Figure 2 shows the limit parabolas $J_2 = 3p^2$ ($\lambda = 0$, $p = -\sigma_1/3 < 0$) and $J_2 = (3/4)p^2$ ($\lambda \to -\infty$ or $\sigma_1 \to 0+$, $p > 0$), together with the previous compression limit $J_2 = 3p^2$ ($\lambda = 0$, $p = -\sigma_1/3 > 0$), which corresponds here to $\lambda = -2$. The domain between $J_2 = 3p^2$ and $J_2 = (3/4)p^2$ ($p > 0$) is therefore duplicated by the compression and tension configuration, but the tension configuration extends the previous domain from $J_2 = 3p^2 (p > 0)$ to $J_2 = 3p^2$ ($p < 0$).

For BS,

$$p \begin{smallmatrix} \leq 0 \\ \geq 0 \end{smallmatrix} \text{ as } \begin{matrix} 0 \geq \lambda \geq -1 \\ \lambda \leq -1 \end{matrix}, \quad \frac{d\nu}{d\lambda} < 0 \text{ as } \begin{matrix} 0 \geq \lambda > -1 \\ \lambda < -1 \end{matrix} \ , \tag{25}$$

and

$$\nu(\lambda = 0) = 3, \qquad \nu(\lambda \to -1) \to \infty, \qquad \nu(\lambda \to -\infty) = 3 \quad . \tag{26}$$

The limit parabolas $J_2 = 3p^2$ ($p < 0$, $p > 0$) are shown in Figure 3, so the previous domain is extended to $J_2 = 3p^2$ ($p < 0$) with no duplication. <u>However, the domain between $J_2 = 0$ and $J_2 = 3/4p^2$ is not covered by either of the compression or tension configurations, and will therefore require a loading configuration combining shear stress and confining pressure (unless the BS configuration is possible with both $\sigma_1 > 0$, $\sigma_2 > 0$).</u>

<u>References</u>

M2, M3 Shell reports prepared by L. W. Morland.

Spring, U. and Morland, L. W. (1981), Viscoelastic Solid Relations for the Deformation of Ice, Cold Reg. Sci. Tech. (submitted).

Morland, L. W. (1980), Constitutive Laws for Ice, Cold Reg. Sci. Tech., <u>1</u>, 101-108.

Fig. 1 TIS and BS invariants domain for compression tests, $\sigma<0$, $\lambda<0$.

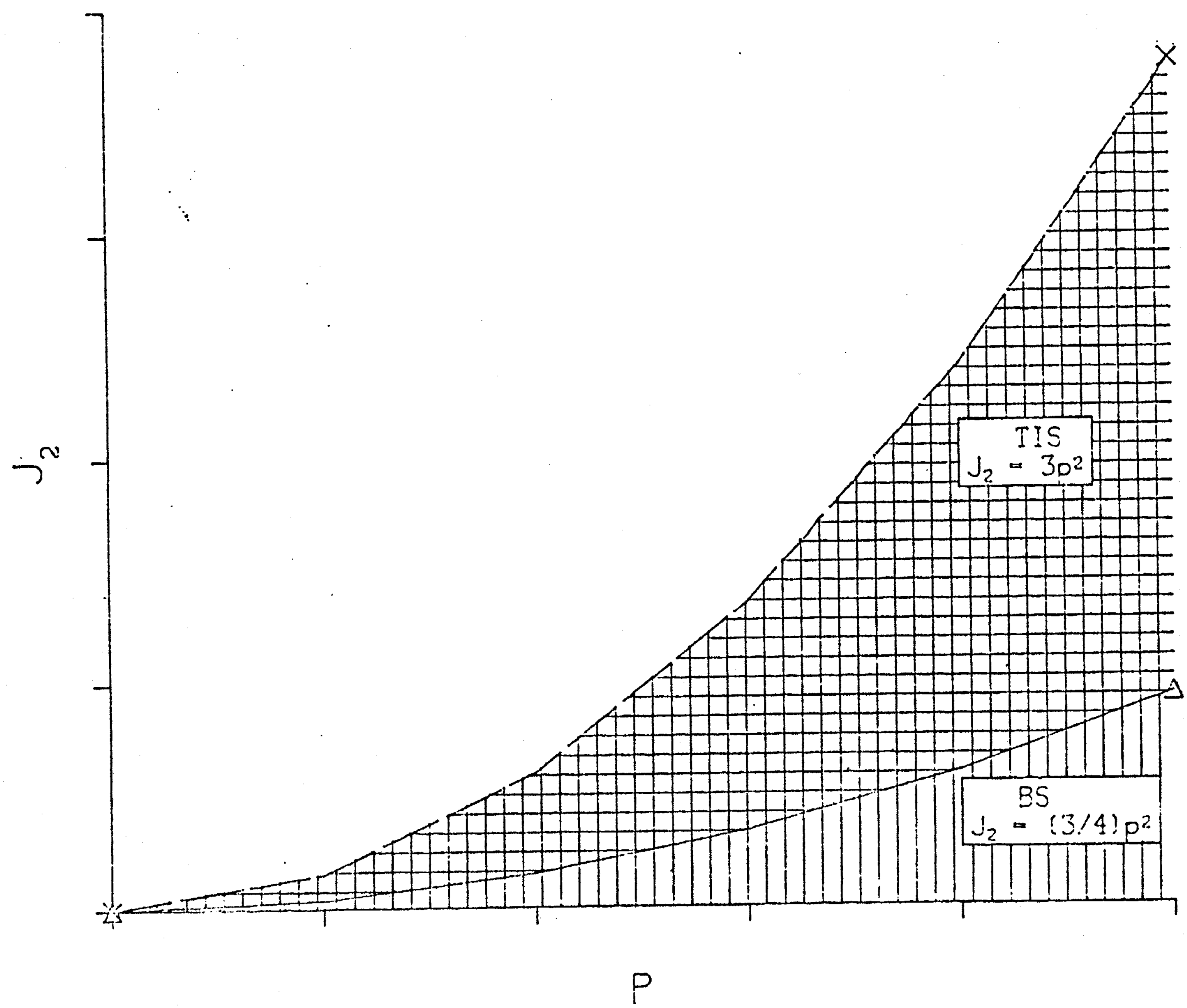


Fig. 2 TIS in axial tension($\lambda<0$).

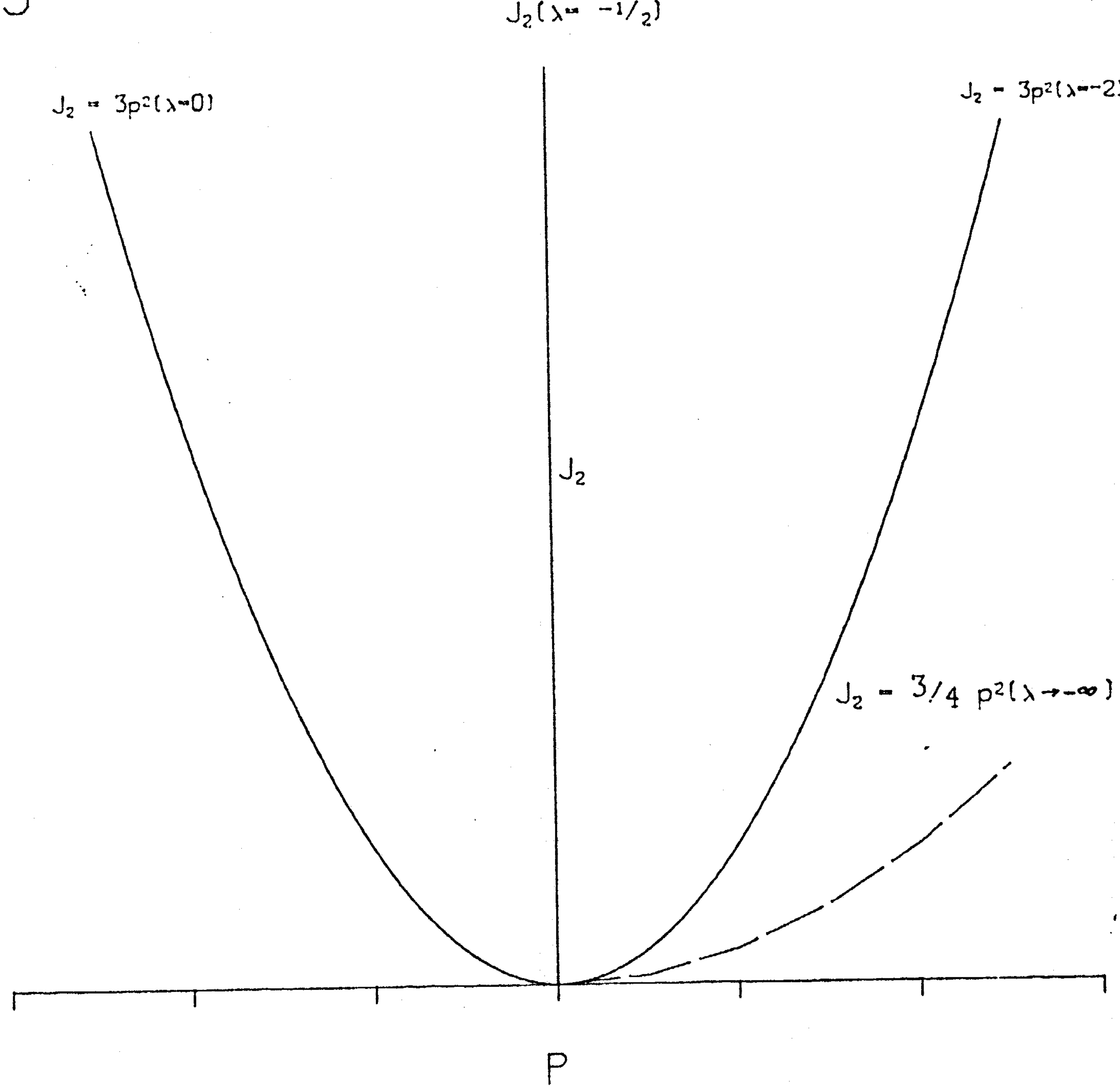
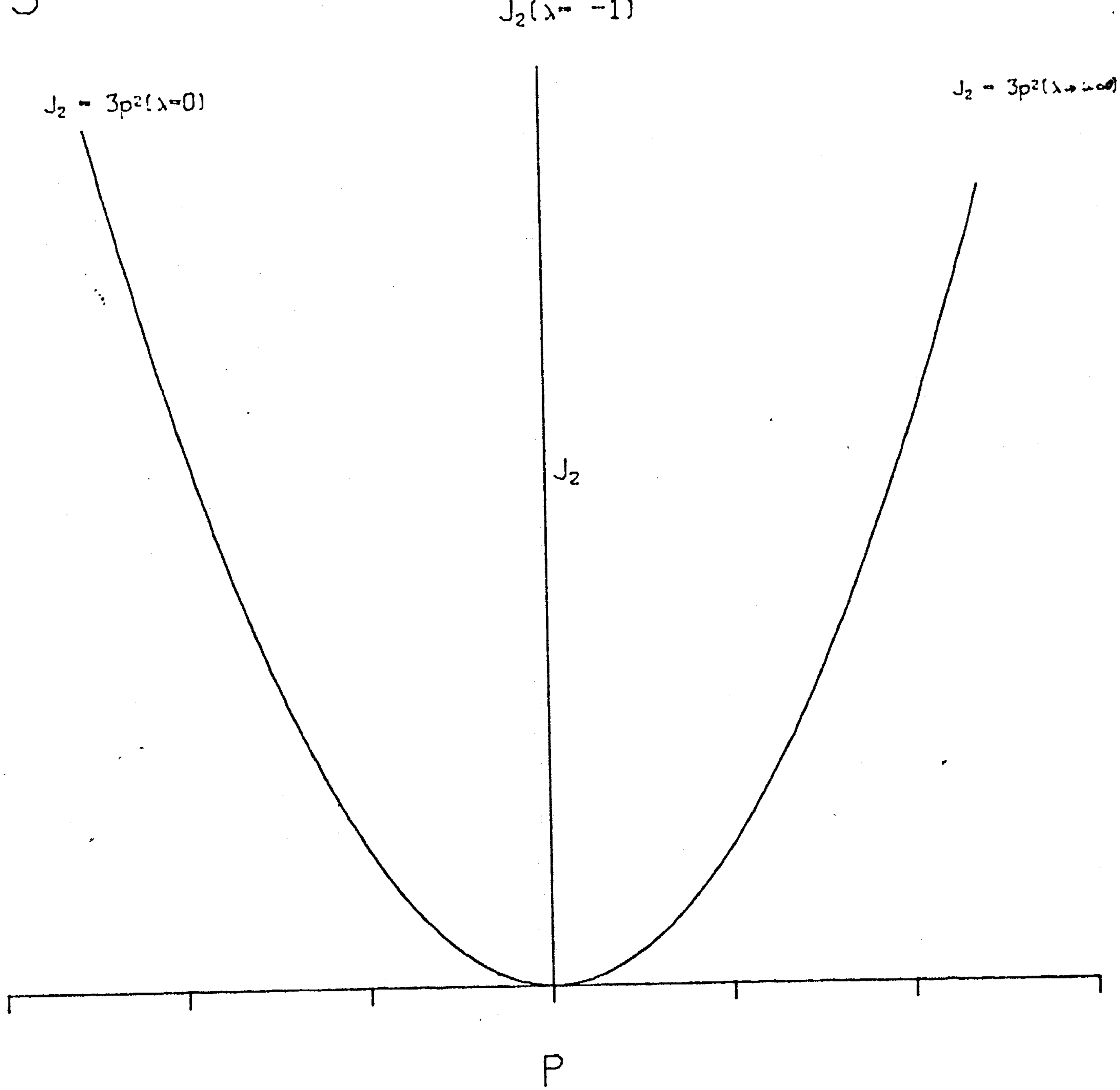


Fig. 3 BS in axial tension(λ<0).



MECHANICAL PROPERTIES OF SEA ICE

Report M6: Constant Stress-Rate and Constant-Strain Tests

1. Introduction

The minimal differential operator viscoelastic solid relation referred to an isotropic configuration, with the incompressibility approximation, involves five response coefficients, each in general functions of two stress and two strain invariants and their rates (Spring and Morland, 1981). In uniaxial stress, two strain-rate coefficients occur only in a particular combination, as do two strain coefficients, so only those independent functions of invariants appear in uniaxial stress relations. This is also the situation in a transversely isotropic stress configuration (TIS), so that a biaxial stress configuration (BS) is required to distinguish the five functions. The stress invariants domains for TIS and BS are described in Reports M2 and M5.

Focusing on uniaxial stress tests, there are three functions to determine, each a function of one stress and one strain invariant, and their rates, which may be combinations of the principal invariants. Only one stress and one strain invariant can be distinguished in the uniaxial stress geometry. Conventional constant load and constant displacement rate tests determine two relations between the three functions (usually restricted to a compression domain), so that a full uniaxial stress description needs a further independent relation. Constant load rate and constant displacement test responses are now analyzed on the assumption that the response coefficients do not depend on invariant rates and it is shown that neither provides an independent relation between the coefficients. Thus, only two of the above four tests provide independent relations for the three response coefficients describing uniaxial stress response in the model. The implication is discussed further in the concluding remarks.

It is convenient to introduce the following terminology:

CL = constant load (constant nominal stress),

CLR = constant load rate,

CD = constant displacement (constant engineering strain),

CDR = constant displacement rate.

Corresponding constant stress and constant strain rate tests refer to current configuration measures: traction per unit current area (Cauchy stress) and the symmetric part of the spatial velocity gradient.

2. Uniaxial Stress Response

From Spring and Morland (1981), equation (28),

$$(1-e)^3\,\bar{\sigma} + \hat{\psi}\,(1-e)^2\,[(1-e)\dot{\bar{\sigma}} - 2\dot{e}\bar{\sigma}] = \frac{3}{2}\,\hat{\phi}\,(1-e)\,\dot{e} + \hat{\omega}\,e, \tag{1}$$

where $\bar{\sigma}$ is the axial compressive nominal stress, e is the axial contraction per unit initial length, and the three response coefficients $\hat{\psi}$, $\hat{\phi}$, $\hat{\omega}$ are functions of $\bar{\sigma}$, e; that is, independent of $\dot{\bar{\sigma}}$ and $\dot{e}$. A possible dependence on the square of the strain-rate term has been ignored, since for sensible stress jump-strain jump relations its coefficient must be rate-dependent and vanish as rates become infinite. If included, the $\hat{\phi}\,\dot{e}$ term becomes more complicated, but only a single combination function replaces $\hat{\phi}$. It is supposed that the response coefficients $\hat{\psi}$, $\hat{\phi}$, $\hat{\omega}$ are bounded, in particular as $\bar{\sigma}$ or $e \to 0$. The stress invariants are

$$p = \frac{1}{3}\,(1-e)\,\bar{\sigma},\quad J_2 = \frac{1}{3}\,(1-e)^2\,\bar{\sigma}^2,\quad J_3 = -\frac{2}{27}\,(1-e)^3\,\bar{\sigma}^3, \tag{2}$$

where p is the mean pressure and J_2 and J_3 are principal deviatoric invariants. Note that $\sigma = (1-e)\bar{\sigma}$ where σ is the Cauchy compressive axial stress. The strain invariants are

$$K_1 = (1-e)^2 + 2(1-e)^{-1},\qquad K_2 = 2(1-e) + (1-e)^{-2}, \tag{3}$$

with

$$K_1 = K_2 = 3(1+e^2) + 0\,(e^3) \tag{4}$$

in a small strain approximation. Clearly, dependence on $\bar{\sigma}$ does not distinguish dependence on J_2 and J_3 (nor on J_2 and p), and dependence on e does not distinguish dependence on K_1 and K_2, which in fact are identical to $0(e^2)$.

Assuming that the strain response $e(t)$ to constant load $\bar{\sigma}$ is monotonic, t can be eliminated in terms of e, and the strain-rate $\dot{e}$ expressed as a function of $\bar{\sigma}$, e. Thus, from Spring and Morland (1981), equation (33),

$$\text{CL:} \quad \dot{\bar{\sigma}} = 0 \ , \qquad \dot{e} = F\ (\bar{\sigma},\ e)\ , \qquad e\ (0) = e_e\ (\bar{\sigma}), \tag{5}$$

where $e_e\ (\bar{\sigma})$ is the initial (elastic) strain jump to a stress jump $\bar{\sigma}$, and

$$F\ (\bar{\sigma},\ e) = \frac{(1-e)^3\ \bar{\sigma} - \hat{\omega} e}{\frac{3}{2}\hat{\phi}(1-e)+2\hat{\psi}\bar{\sigma}(1-e)^2}\ . \tag{6}$$

At constant strain-rate $\dot{e} = w$, it is supposed that the family of response curves $\bar{\sigma}(t)$ for different w do not intersect, so that there is a monotonic $\bar{\sigma} - w$ relation at each e which can be inverted:

$$\text{CDR:} \quad \dot{e} = w = \text{constant}\ , \qquad w = W\ (\bar{\sigma},\ e)\ , \qquad \bar{\sigma} = G\ (w,\ e). \tag{7}$$

From the response family $G\ (w,e)$, a family of generalised Young's moduli

$$E\ (\bar{\sigma},\ e) = \frac{\partial G}{\partial e} \tag{8}$$

can be defined, where w is eliminated by (7). Then

$$\text{CDR:} \quad \dot{e} = w\ , \qquad E\ (\bar{\sigma},\ e) = \hat{E}\ (\bar{\sigma},\ e)\ \left\{1 - \frac{F\ (\bar{\sigma},\ e)}{W\ (\bar{\sigma},\ e)}\right\}, \tag{9}$$

where

$$\hat{E}\ (\bar{\sigma},\ e) = \frac{3\hat{\phi}}{2(1-e)^2\hat{\psi}} + \frac{2\bar{\sigma}}{1-e}\ . \tag{10}$$

Thus, given the response functions $E\ (\bar{\sigma},\ e)$, $W(\bar{\sigma},\ e)$ from CDR tests, and $F\ (\bar{\sigma},\ e)$ from CL tests, (6) and (10) are two relations for the three response coefficients. Models constructed by Spring and Morland (1981) from idealized CL responses and considering constant $\hat{E}$ and $\hat{E}$ with rapid exponential decay in e showed that the CDR response $G\ (w,e)$ was insensitive to the choice of $\hat{E}$. No conclusion can be drawn about the sensitivity of the combination $\hat{E}$ (10) to changes in G, hence E, F and W.

Now consider constant load-rate responses e(t) at different load-rates $\dot{\bar{\sigma}} = q$. Assuming that a family of non-intersecting curves are obtained as q varies, illustrated in Figure 1(a), with the time to reach a given strain e increasing as the rate q increases (in accord with general observation), then the corresponding curves for e as a fraction of $\bar{\sigma} = qt$ fan out, as shown in Figure 1(b), and hence remain non-intersecting. Thus, at each $\bar{\sigma}$, there is a monotonic e-q relation which can be inverted:

$$\text{CLR:} \quad \dot{\bar{\sigma}} = q = \text{constant}, \quad q = Q\,(\bar{\sigma}, e), \quad e = e^*(\bar{\sigma}, q), \tag{11}$$

with the following properties

$$\frac{\partial e^*}{\partial \bar{\sigma}} > 0, \quad \frac{\partial e^*}{\partial q} < 0, \quad \frac{\partial Q}{\partial \bar{\sigma}} > 0, \quad \frac{\partial Q}{\partial e} < 0. \tag{12}$$

Using (11), the strain-rate can be expressed in terms of $\bar{\sigma}$, e:

$$\dot{e} = q\,\frac{\partial e^*}{\partial \bar{\sigma}} = D\,(\bar{\sigma}, e). \tag{13}$$

Substituting in (1) for CLR:

$$D\,(\bar{\sigma}, e) = \frac{(1-e)^3\,\bar{\sigma} - \hat{\omega} e + (1-e)^3\,\hat{\psi} Q\,(\bar{\sigma}, e)}{\frac{3}{2}\,\hat{\phi}\,(1-e) + 2\,\hat{\psi}\,\bar{\sigma}\,(1-e)^2} \tag{14}$$

By (10)

$$\frac{3}{2}\,\hat{\phi}\,(1-e) + 2\,\hat{\psi}\,\bar{\sigma}\,(1-e)^2 = (1-e)^3\,\hat{\psi}\,\hat{E}\,(\bar{\sigma}, e), \tag{15}$$

and using (15) and (6), (14) becomes

$$D\,(\bar{\sigma}, e) = F\,(\bar{\sigma}, e) + \frac{Q\,(\bar{\sigma}, e)}{\hat{E}\,(\bar{\sigma}, e)}\;, \tag{16}$$

which is independent of the response coefficients, and hence a consistency check on the functions D and Q given by the CLR test, and in turn on the adopted model. Thus the CLR response functions D and Q do not determine a third independent relation for $\hat{\phi}$, $\hat{\psi}$, $\hat{\omega}$. Any two of the CL, CDR, and CLR tests, giving the relations (6), (10), (14), are independent, but a third independent response is still required to complete the uniaxial description of the model.

Turning to constant displacement responses $\bar{\sigma}$ (t) at different constant strains e, then assuming these are monotonic and non-intersecting, there is a unique relation $t = T(\bar{\sigma}, e)$ by which t can be eliminated. It is expected that $\dot{\bar{\sigma}} \leq 0$ at constant strain, so that the response is defined by

$$\text{CD:} \quad \dot{e} = 0, \quad \dot{\bar{\sigma}} = -L(\bar{\sigma}, e) \leq 0 \ (L \geq 0). \tag{17}$$

Substituting in (1),

$$L(\bar{\sigma}, e) = \frac{(1-e)^3 \bar{\sigma} - \hat{\omega} e}{(1-e)^3 \hat{\psi}} \tag{18}$$

Hence, by (6) and (15),

$$L(\bar{\sigma}, e) = \hat{E}(\bar{\sigma}, e) F(\bar{\sigma}, e), \tag{19}$$

again, independent of the response coefficients, and only a consistency check on the model. No independent relation for the coefficients is provided.

3. Concluding Remarks

Thus, not only does the uniaxial stress response (1) involve too many functions to be determined by CL and CDR tests, but the other fundamental tests, CLR and CD, appropriate to the derivatives occuring in (1), provide no further information, and are only consistency checks on the model assumptions. Any two of these four tests provide the independent information, and can be selected for practical convenience. An explicit model therefore must eliminate one of the response coefficients. Elastic jump relations in the parent tensor relation require both tensor rate terms, that is, $\psi \neq 0$, $\phi \neq 0$, so $\hat{\omega} = 0$ would be a first tentative choice, eliminating dependence on the strain tensor. However, dependence of ψ, ϕ on the strain invariants still implies dependence on the reference configuration, and hence induced anisotropy from preloaded states (Spring and Morland 1981). Consider

$$\hat{\omega} = 0, \tag{20}$$

then CL and CDR, (6) and (15), give

$$\hat{\psi} = \frac{\bar{\sigma}}{F\hat{E}}, \quad \hat{\phi} = \frac{2\bar{\sigma}(1-e)}{3F\hat{E}} \left\{(1-e)\,\hat{E} - 2\bar{\sigma}\right\}, \tag{21}$$

while CL and CLR, (6) and (14), give

$$\hat{\psi} = \frac{\bar{\sigma}\,(D-F)}{FQ}, \quad \hat{\phi} = \frac{2\,\bar{\sigma}\,(1-e)}{3FQ} \left\{(1-e)\,Q - 2\,\bar{\sigma}\,(D-F)\right\}, \tag{22}$$

CL and CD, (6) and (18), give

$$\hat{\psi} = \frac{\bar{\sigma}}{L}, \quad \hat{\phi} = \frac{2\,\bar{\sigma}\,(1-e)}{3FL} \left\{(1-e)\,L - 2\,\bar{\sigma}\,F\right\}. \tag{23}$$

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1. Spring, U. and Morland, L. W. (1981), as in M5.

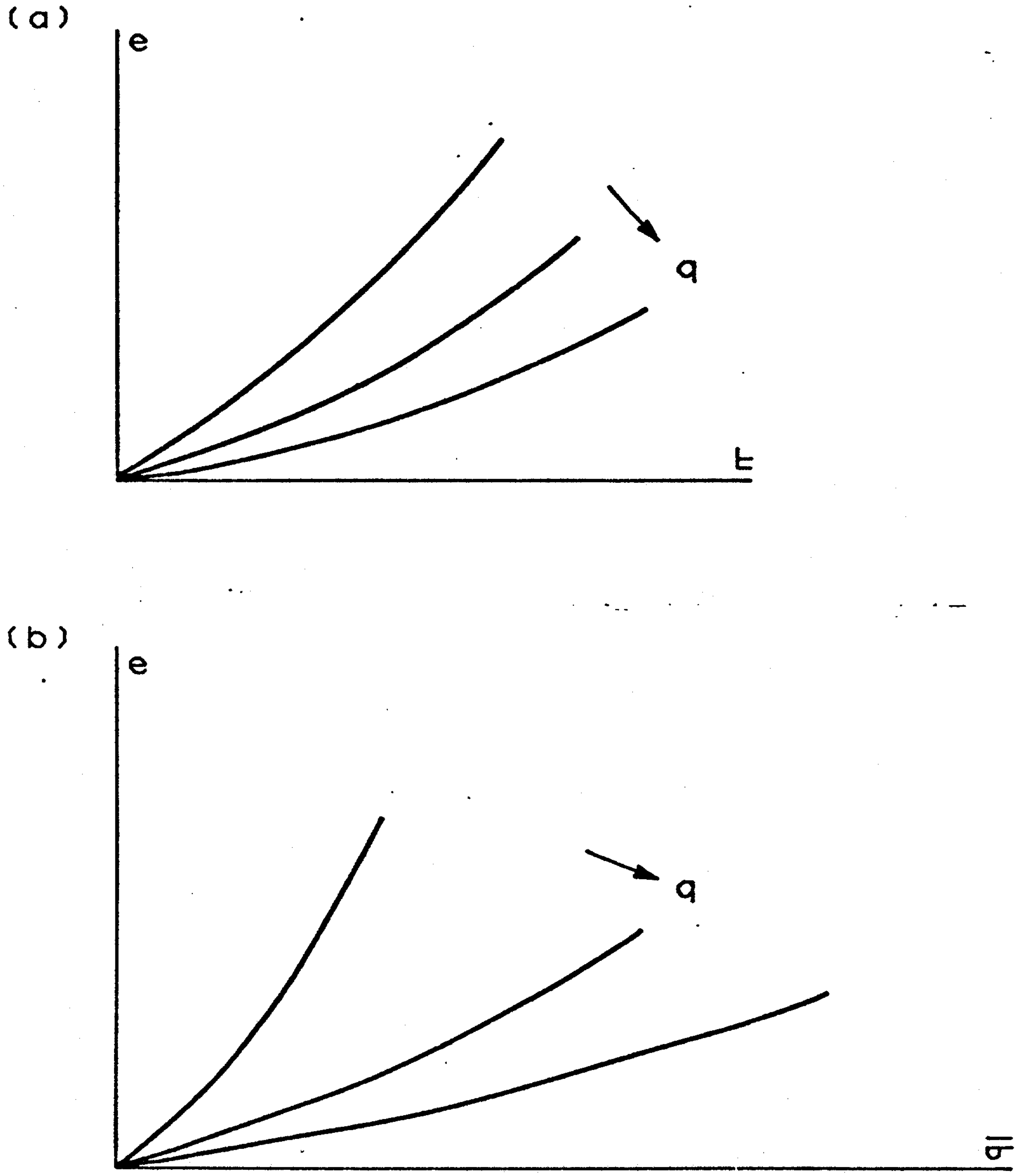

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Figure 1

MECHANICAL PROPERTIES OF SEA ICE

Report M7: Uniaxial Stress Data Correlation

1. Introduction

The differential operator viscoelastic solid relation which assumes incompressibility and isotropy in the reference configuration requires strain, strain-rate, stress and stress-rate terms in order to describe observed uniaxial stress behavior at constant stress and at constant strain-rate (Spring and Morland 1981). A frame indifferent tensor relation which is linear in the adopted stress-rate tensor and allows linear and quadratic dependence on the strain tensor and strain-rate tensor involves five response coefficients which are functions of the various tensor invariants. In uniaxial stress, or transversely isotropic stress (TIS, Report M5), there is only one independent relation, and this involves three independent response coefficients. The two strain terms and the two strain-rate terms form composite strain and strain-rate terms respectively. Uniaxial stress response is therefore governed by three response coefficients and can determine, in principle, three such coefficients as functions of one stress component and one strain component (not two independent deviatoric stress invariants and two independent strain invariants). The model assumes no dependence of the response coefficients on invariant rates.

Report M6 describes four fundamental test configurations: constant load (CL), constant displacement rate (CDR), constant load rate (CLR), and constant displacement (CD), and shows that only two of the four provide independent relations between the response coefficients. That is, the data functions determined by the four tests satisfy various identities for this model, which means that the test responses reflect common properties of the model. The next question is whether a third relation can be determined by other uniaxial stress loading histories, including a sequence of applied stress jumps to correlate with the jump relations given by the model. It is now shown that uniaxial stress response is described by two response functions only, and can therefore determine only two functions; that is, these are at most, two independent test responses. The uniaxial relation can be expressed in terms of functions from various pairs of the above four tests.

An immediate conclusion is that the three separate response coefficients required in the tensor relation, even as functions only of one stress and one strain, cannot be determined by uniaxial stress response. It has not yet been shown that multiaxial test data which determines the response over a domain of two stress and two strain invariants will yield sufficient independent relations to determine the required five response coefficients of this model, or whether the uniaxial response defficiency extends also to multiaxial response. Since the five associated terms are introduced to represent distinct directional and rate response features, they are expected to be predicted by general multiaxial loading tests, but if not the apparently general relation must be reducible to a form involving fewer coefficients. The reduction obtained by setting the three uniaxial coefficients zero in turn are examined, and it is shown that the reduced model is physically sensible in only one case, namely, when the strain-tensor term is absent. This implies that the solid exhibits no creep relaxation; following an elastic strain jump, on complete unloading from any stress-strain configuration (Spring and Morland 1981). Finally, the stress-strain domains covered by the four tests above are noted to show that they do not extend to domains which can occur in quite simple tension - compression loading.

2. Uniaxial Stress Relations

The uniaxial stress relation (Report M6) is

$$(1-e)^3\ \bar{\sigma} + \hat{\psi}\ (1-e)^2\ [(1-e)\ \dot{\bar{\sigma}} - 2\dot{e}\bar{\sigma}] = \frac{3}{2}\ \hat{\phi}\ (1-e)\dot{e} + \hat{\omega}e, \tag{1}$$

where the strain-rate coefficient $\phi\ (\bar{\sigma}, e) = \phi_1$ in the tensor relation (Spring and Morland 1981) when $\phi_2 = 0$ is adopted, and the strain coefficient $\hat{\omega}\ (\bar{\sigma}, e)$ is a combination of ω_1 and ω_2. The simplification $\phi_2 = 0$ is necessary to derive the jump relation as a limit of continuous changes satisfying the differential operator law when rate dependence is excluded from the coefficients. In a constant load test the measured response is

$$\text{CL:}\ \dot{\bar{\sigma}} = 0, \quad \dot{e} = F\ (\bar{\sigma}, e)\ , \quad e \geq e\ (0) = e_e\ (\bar{\sigma}); \tag{2}$$

in a constant displacement rate test

$$\text{CDR:}\ \dot{e} = w\ , \quad \frac{d\sigma}{de} = \hat{E}\ (\bar{\sigma},\ e)\ \left\{1 - \frac{F\ (\bar{\sigma},\ e)}{w}\right\}, \quad w = W\ (\bar{\sigma},\ \dot{e}); \tag{3}$$

in a constant load rate test

$$\text{CLR:}\ \dot{\bar{\sigma}} = q\ , \quad \dot{e} = D\ (\bar{\sigma},\ e)\ , \quad q = Q\ (\bar{\sigma},\ e); \tag{4}$$

and in a constant displacement test

$$\text{CD:}\ \dot{e} = 0, \quad \dot{\bar{\sigma}} = -\ L\ (\bar{\sigma},\ e). \tag{5}$$

The measured functions F, $\hat{E}$ (using W), D, Q, L are related to the response coefficients by

$$F = \frac{(1-e)^3\ \bar{\sigma} - \hat{\omega}e}{\frac{3}{2}\ \hat{\phi}\ (1-e) + 2\ \hat{\psi}\ \bar{\sigma}\ (1-e)^2}\ , \quad \hat{E} = \frac{3\hat{\phi}}{2\ \hat{\psi}\ (1-e)^2} + \frac{2\bar{\sigma}}{1-e}$$

$$D = \frac{(1-e)^3\ \bar{\sigma} - \hat{\omega}e + (1-e)^3\ \hat{\psi}\ Q}{\frac{3}{2}\ \hat{\phi}\ (1-e) + 2\ \hat{\psi}\ \bar{\sigma}\ (1-e)^2}\ , \quad L = \frac{(1-e)^3\ \bar{\sigma} - \hat{\omega}e}{(1-e)^3\ \hat{\psi}} \tag{6}$$

which imply the identities

$$D = F + \frac{Q}{\hat{E}}\ , \quad L = \hat{E}\ \dot{F}\ , \tag{7}$$

independent of the reponse coefficients

Using the definitions (6), the general uniaxial relation (1) can be expressed solely in terms of the measured functions from any pair of the six possible pairs of tests:

$$\text{CL and CDR:} \quad \dot{\bar{\sigma}} = \hat{E}\ (\dot{e}-F), \tag{8}$$

$$\text{CL and CLR:} \quad (D-F)\ \dot{\bar{\sigma}} = Q\ (\dot{e}-F), \tag{9}$$

$$\text{CDR and CLR:} \quad \dot{\bar{\sigma}} - Q = \hat{E}\ (\dot{e} - D), \tag{10}$$

$$\text{CL and CD:} \quad F\ \dot{\bar{\sigma}} = L\ (\dot{e} - F), \tag{11}$$

$$\text{CDR and CD:} \quad \dot{\bar{\sigma}} + L = \hat{E}\ \dot{e}, \tag{12}$$

$$\text{CLR and CD:} \quad D\ \dot{\bar{\sigma}} = (L + Q)\ \dot{e} - LD\ . \tag{13}$$

That is, the response to arbitrary uniaxial stress loading is governed by any one of the six equivalent relations (8) - (13), so that measured response to any alternative test loading gives information only in terms of the above sets of functions, and on no other combination of the response coefficients. Furthermore, by (8), an infinitesimal strain jump relation

$$e = \hat{h}\ (\bar{\sigma}, \bar{\sigma}_o, e_o), \quad \hat{h}\ (\bar{\sigma}_o, \bar{\sigma}_o, e_o) = 0, \tag{14}$$

is given by

$$\frac{\partial \hat{h}}{\partial \bar{\sigma}} = \frac{1}{\hat{E}\ (\bar{\sigma}, e_o)}\ , \tag{15}$$

so that a sequence of stress jump tests can determine only $\hat{E}\ (\bar{\sigma}, e)$ again. Thus, the three response coefficients of (1) derived from the tensor relation cannot be determined by uniaxial stress response, which is governed solely by two response functions. That is, the five coefficients of the tensor relation reduce to two combination functions for uniaxial stress, and not the apparent three shown in (1).

3. Two Response Coefficients

Two independent sets of test data will determine two response coefficients. Consider the reduced models obtained by setting $\hat{\phi}$, $\hat{\psi}$, $\hat{\omega}$ zero in turn.

If $\hat{\phi} = 0$, then by (6)

$$\hat{\phi} = 0: \quad \hat{E} = \frac{2\bar{\sigma}}{1-e} = 0\ (\bar{\sigma})\ , \ \hat{E} = 0 \text{ at } \bar{\sigma} = 0. \tag{16}$$

Now $\hat{E}$ is a modulus >> $\bar{\sigma}$, and in particular the jump relation (15) implies that $\partial\hat{h}/\partial\bar{\sigma} = O(\bar{\sigma}^{-1})$ as $\bar{\sigma} \to 0$ which is not compatible with an infinitesimal strain jump when a stress jump from $\bar{\sigma} = 0$ is applied. Thus a model which relates stress, stress-rate, and strain is not acceptable.

If $\hat{\psi} = 0$, when the model relates stress, strain and strain-rate, then $\hat{E}$ is infinite and there is no strain jump if a stress jump is applied. This is an acceptable approximation since strain jumps are small compared with creep strains in many applications. The test functions F and D remain <u>bounded</u>, form (6), but are identical by (7); Q is a <u>bounded</u> quantity by definition. Thus by (9), $\dot{e} = F$ for all $\bar{\sigma}(t)$, not just $\bar{\sigma}$ = constant, and in the CD test, direct form (1),

$$\text{CD:} \quad \dot{e} = 0, \quad (1-e)^3\,\bar{\sigma} = \hat{\omega}e, \tag{17}$$

which implies that the stress remains constant, also unacceptable.

If $\hat{\omega} = 0$, there is no dependence on the strain tensor, but there is dependence on strain through dependence of ϕ_1, ψ on strain invariants, hence dependence of $\hat{\phi}$, $\hat{\psi}$ on e. Now, by (6),

$$F\,(0,\ e) = 0, \tag{18}$$

and on complete unloading from any stress-strain state there is no subsequent creep relaxation ($\dot{e} = 0$) following the elastic jump. In classical linear viscoelastic solid models, which exhibit decreasing strain-rate in time under constant stress and satisfy a superposition principle, there is always relaxation on full or partial unloading ($\dot{e} < 0$). It is not clear from the ice-mechanics literature whether such a nonrelaxing property (on complete unloading) is a reasonable approximation, but this reduced model is otherwise physically sensible. Report M6 displayed various expressions for $\hat{\phi}$, $\hat{\psi}$ in terms of F, $\hat{E}$ D, Q, L when $\hat{\omega} = 0$. It is clear from (6) that complete unloading from a constant stress state in which $\dot{e} = F > 0$, can produce relaxation $\dot{e} = F\,(0,\ e) < 0$ only if $\hat{\omega}$ increases significantly with such decrease of stress and infinitesimal elastic strain decrease. Relaxation after any small stress decrease, analogous to the classical linear model, would imply more dramatic increase of $\hat{\omega}$ so that $\hat{\omega}$ derivatives would be large in some sense. Unloading response must be investigated to find how much relaxation occurs; this model does not appear to be satisfactory if relaxation occurs on any small partial unloading.

4. Stress-Strain Domains

In the constant load test with a load range $0 < \bar{\sigma} < \bar{\sigma}^*$, $F(\bar{\sigma}, e)$ is defined only for $e \geq e_e(\bar{\sigma})$, to the strain limit covered by the test. That is, F is not defined in the domain $0 \leq e < e_e(\bar{\sigma})$, where $e_e(\bar{\sigma})$ is the elastic strain jump for an applied jump stress $\bar{\sigma}$, and $e_e(0) = 0$.

In the constant displacement rate test to measure $\hat{E}(\bar{\sigma}, e)$, there is again a limit stress-strain curve $e_w(\bar{\sigma})$ which depends on the maximum $\dot{e} = w$ applied. Its initial slope is $\hat{E}(0, 0)$, since $F(0, 0) = 0$, for all $w > 0$, which is a restriction of the model. Now $e_e'(0) = \hat{E}^{-1}(0, 0)$, so the two limit curves coincide as $\bar{\sigma} \to 0$.

In the constant load-rate test, see Figure 1 (b) of Report M6, $D(\bar{\sigma}, e)$, $Q(\bar{\sigma}, e)$ are determined only in a domain $e > e_q(\bar{\sigma})$ where q is the maximum $\dot{\bar{\sigma}}$ applied, and as $\bar{\sigma} \to 0$, by (7), the limit slope $e_q'(0) = D/Q = \hat{E}^{-1}(0,0)$ again.

In the constant displacement test the limit relaxation curve $\bar{\sigma} = \bar{\sigma}_D(t)$ for the $L(\bar{\sigma}, e)$ domain depends on the maximum strain e applied with $\bar{\sigma}_D(0)$ given by the initial jump condition. For each constant e, the upper stress limit is $\bar{\sigma}_D(0)$, and hence the domain covered is precisely $e \geq e_e(\bar{\sigma})$ again.

While the data functions will cover all uniaxial compression loading (presumably extended to some "compression" domain in multiaxial stress), the excluded domain $0 \leq e \leq e_e(\bar{\sigma})$ can be reached by loading histories reaching a current compression state. Consider a tension-compression history:

$$\begin{aligned} 0 < t < t_1 &: \quad \bar{\sigma} = -T < 0 \text{ (tension)} \\ t > t_1 &: \quad \bar{\sigma} = P > 0 \text{ (compression).} \end{aligned} \tag{19}$$

During the tension stage there will be an elastic tension jump followed by tensile creep, resulting in $e = e_1 < 0$ at $t = t_1$. Now the stress jump P + T at $t = t_1$ causes an elastic compression jump followed by compressive creep. If P is sufficiently large (compared with T), the net creep must eventually become positive, with $\bar{\sigma} = P > 0$, so lie in the domain $0 \le e \le e_e\,(P)$. This domain can only be reached by tension-compression tests.

MECHANICAL PROPERTIES OF SEA ICE

Report M8. Reduced Viscoelastic Solid Models

1. Introduction

The isotopic viscoelastic solid relation discussed in earlier reports has the form

$$\begin{aligned} &\underset{\sim}{S} + \psi\, [\dot{\underset{\sim}{S}} + \underset{\sim}{S}\, (\underset{\sim}{D}+\underset{\sim}{W}) + (\underset{\sim}{D}-\underset{\sim}{W})\, \underset{\sim}{S} - \tfrac{2}{3}\, \mathrm{tr}\, (\underset{\sim}{S}\underset{\sim}{D})\, \underset{\sim}{1}] \\ &\quad = \phi \underset{\sim}{D} + \omega_1\, [\underset{\sim}{B} - \tfrac{1}{3} K_1\, \underset{\sim}{1}] + \omega_2\, [\underset{\sim}{B}^2 - \tfrac{1}{3}\, (K_1^2 - 2K_2)\, \underset{\sim}{1}], \end{aligned} \tag{1}$$

involving four response coefficients ψ, ϕ, ω_1, ω_2. The stress and stress-rate tensors are assumed to occur only as a linear combination, and a possible frame indifferent term in $\underset{\sim}{D}^2$ is eliminated to ensure bounded elastic jump relations in the limit of increasingly fast stress changes. Since $\psi \neq 0$ is required to recover observed uniaxial response at constant strain-rate, there is an equivalent alternative relation

$$\begin{aligned} &\dot{\underset{\sim}{S}} + \underset{\sim}{S}\, (\underset{\sim}{D} + \underset{\sim}{W}) + (\underset{\sim}{D} - \underset{\sim}{W})\, \underset{\sim}{S} - \tfrac{2}{3}\, \mathrm{tr}\, (\underset{\sim}{S}\underset{\sim}{D})\, \underset{\sim}{1} + \psi^*\, \underset{\sim}{S} \\ &\quad = \phi^*\, \underset{\sim}{D} + \omega_1^*\, [\underset{\sim}{B} - \tfrac{1}{3} K_1\, \underset{\sim}{1}] + \omega_2^*\, [\underset{\sim}{B}^2 - \tfrac{1}{3}\, (K_1^2 - 2K_2)\, \underset{\sim}{1}]. \end{aligned} \tag{2}$$

The previous analysis supposes that ψ, ϕ, ω_1, ω_2 are functions of all stress and strain invariants, but not invariant rates, which therefore applies to ψ^*, ϕ^*, ω_1^*, ω_2^*.

In uniaxial compressive (engineering) stress $\bar{\sigma}$, axial strain e, (1) and (2) give the respective single relations

$$\begin{aligned} &(1-e)^3\, \bar{\sigma} + (1-e)^2\, \hat{\psi}\, [(1-e)\, \dot{\bar{\sigma}} - 2\bar{\sigma}\dot{e}] \\ &\quad = \tfrac{3}{2}\, \hat{\phi}\, (1-e)\, \dot{e} + \hat{\omega} e, \end{aligned} \tag{3}$$

and

$$(1-e)^3\,\dot{\bar{\sigma}} - 2\,(1-e^2)\,\bar{\sigma}\,\dot{e} + (1-e)^3\,\hat{\psi}^*\,\bar{\sigma}$$
$$= \frac{3}{2}\,\hat{\phi}^*\,(1-e)\,\dot{e} + \hat{\omega}^* e, \tag{4}$$

where $\hat{\psi}$, $\hat{\phi}$, $\hat{\omega}$, $\hat{\psi}^*$, $\hat{\phi}^*$, $\hat{\omega}^*$, are functions of $(\bar{\sigma}, e)$ and ω and ω^* are composite functions of ω_1, ω_2 and ω_1^*, ω_2^*, respectively. Both (3) and (4) can be expressed in the common form

$$\dot{\bar{\sigma}} = \hat{E}\,(\dot{e}-F), \tag{5}$$

where

$$\hat{E} = \frac{3\hat{\phi}}{2\,\hat{\psi}\,(1-e)^2} + \frac{2\bar{\sigma}}{1-e} = \frac{3\hat{\phi}^*}{2\,(1-e)^2} + \frac{2\bar{\sigma}}{1-e}, \tag{6}$$

$$F = \frac{(1-e)^3\,\bar{\sigma} - \hat{\omega}e}{\frac{3}{2}\,\hat{\phi}\,(1-e) + 2\,\hat{\psi}\,\bar{\sigma}\,(1-e)^2} = \left[\hat{\psi}^*\bar{\sigma} - \frac{\hat{\omega}^* e}{(1-e)^3}\right]\Big/\hat{E}, \tag{7}$$

involving only two data functions $F\,(\bar{\sigma}, e)$, $\hat{E}\,(\bar{\sigma}, e)$ determined by constant load and constant displacement-rate tests. While (3) and (4) are equivalent when all response coefficients are general functions of $(\bar{\sigma}, e)$, making the same restrictions on the dependence of $\hat{\psi}$, $\hat{\phi}$, $\hat{\omega}$ and $\hat{\psi}^*$, $\hat{\phi}^*$, $\hat{\omega}^*$ leads to different shapes of F and $\hat{E}$, and hence different uniaxial response; that is, (3) and (4) become different models.

2. Restricted dependence

Since the general uniaxial relation (5) contains only two functions F and $\hat{E}$, only two functions can be determined by uniaxial tests. Various alternative pairings were exhibited in Report M7, where it was shown also that $\hat{\psi} \neq 0$, $\hat{\phi} \neq 0$ is necessary, while $\hat{\omega} = 0$ is allowed but implies that no creep relaxation occurs when the stress is unloaded from any configuration. For complete unloading at time t_1, with the model (3),

$$t > t_1 : \quad \bar{\sigma} = 0, \quad \dot{e} = F\,(o, e) = -\frac{\hat{\omega}\,(o, e)\,e}{\frac{3}{2}\,\hat{\phi}\,(o, e)(1-e)}, \tag{8}$$

Since relaxation is observed, $\hat{\omega} \neq 0$, and these are three response coefficients to determine from two data functions. Note that measuring the unloading response F (0, e) gives only ratio $\hat{\omega}$ (0, e)/$\hat{\phi}$ (0,e) for $\bar{\sigma} = 0$. Similarly, with (4),

$$t > t_1 : \quad \bar{\sigma} = 0, \quad \dot{e} = F\ (0,\ e) = -\frac{\hat{\omega}^*\ (0,\ e)\ e}{(1-e)^3\ \hat{E}\ (0,\ e)}, \tag{9}$$

and $\omega^* \neq 0$ for relaxation. In both (8) and (9), the creep relaxation is given by

$$t > t_1 : \quad \dot{e} = F\ (0,\ e) = -f\ (e), \quad e\ (t_1) = e_1\ , \tag{10}$$

Where the strain-rate $-f$ (e) depends only on e, but the initial strain e_1 depends on the previous load history, and elastic strain decrease on unloading at time t_1.

If the unloaded response f (e) is known, it gives an independent relation on $\hat{\omega}$, $\hat{\phi}$ if the ratio $\hat{\omega}/\hat{\phi}$ is independent of $\bar{\sigma}$, and an independent relation on $\hat{\omega}^*$, $\hat{E}$ if the ratio $\hat{\omega}^*/\hat{E}$ is independent of $\bar{\sigma}$. First consider

$$\hat{\phi} = \phi_1\ (\bar{\sigma})\ \phi_2\ (e), \quad \hat{\omega} = \phi_1\ (\bar{\sigma})\ \omega_2\ (e), \tag{11}$$

which are the necessary forms for $\hat{\omega}/\hat{\phi}$ independent of $\bar{\sigma}$. Then

$$e\ \omega_2\ (e) = \frac{3}{2}\ \phi_2\ (e)\ f\ (e)\ (1-e) \tag{12}$$

is determined once ϕ_2 (e) is known, and hence $\hat{\omega}$ once $\hat{\phi}$ is known. Alternately

$$\hat{\omega}^* = \omega_1\ (\bar{\sigma})\ \omega_2\ (e), \quad \hat{E} = \omega_1\ (\bar{\sigma})\ E_2\ (e), \tag{13}$$

which requires $\hat{E}$ to have a separable form, gives

$$e\ \omega_2\ (e) = E_2\ (e)\ f\ (e)\ (1-e)^3, \tag{14}$$

so $\hat{\omega}$ is given when $\hat{E}$ is known.

Since elastic strain jumps associated with a stress jump, $\bar{\sigma}$ are infinitesimal,

$$\hat{E} \gg \bar{\sigma} \, , \tag{15}$$

and so from (6),

$$\hat{\phi} \gg \bar{\sigma} \, \hat{\psi} \, , \quad \hat{\phi}^* \gg \bar{\sigma} \, . \tag{16}$$

With these strong inequalities, (6) and (7) give the simpler approximate expressions

$$\hat{E} = \frac{3\hat{\phi}}{2 \, \hat{\psi} \, (1-e)^2} = \frac{3\hat{\phi}^*}{2 \, (1-e)^2} \, , \tag{17}$$

$$F = \frac{(1-e)^3 \, \bar{\sigma} - \hat{\omega} e}{\frac{3}{2} \, \hat{\phi} \, (1-e)} = \frac{(1-e)^3 \bar{\sigma} \, \hat{\psi}^* - \hat{\omega}^* e}{\frac{3}{2} \, \hat{\phi}^* \, (1-e)} \tag{18}$$

While total strain e may be small, it is not necessarily negligible compared to unity, and no simplification is obtained with an approximation $e \ll 1$.

Applying the restricted forms (11), (12), to (17), (18) gives

$$F = - f(e) + F_1 \, (\bar{\sigma}) \, F_2 \, (e) \tag{19}$$

where

$$F_1 \, (\bar{\sigma}) = \frac{\bar{\sigma}}{\phi_1 \, (\bar{\sigma})} \, , \quad F_2 \, (e) = \frac{(1-e)^2}{\frac{3}{2} \, \phi_2 \, (e)} \, , \tag{20}$$

and

$$\hat{E} = \frac{\bar{\sigma}}{F_1 \, (\bar{\sigma}) \, F_2 \, (e) \, \hat{\psi} \, (\bar{\sigma}, \, e)} \tag{21}$$

Thus, $F \, (\bar{\sigma}, \, e)$ must have the particular structure (19), but $\hat{E} \, (\bar{\sigma}, \, e)$ is not restricted since $\hat{\psi}$ is not restricted. Data can be tested to check if (18) is a satisfactory approximation. If (19) holds, (20) and (21) give

$$\phi_1\,(\bar{\sigma}) = \frac{\bar{\sigma}}{F_1\,(\bar{\sigma})}\,,\ \frac{3}{2}\,\phi_2\,(e) = \frac{(1-e)^2}{F_2\,(e)}\,,$$

$$\hat{\psi}\,(\bar{\sigma},\,e) = \frac{\bar{\sigma}}{F_1\,(\bar{\sigma})F_2\,(e)\,\hat{E}\,(\bar{\sigma},\,e)}\,, \tag{22}$$

which, with (12), determine the three response coefficients. A special case is

$$\hat{\phi} = \phi_2\,(e),\quad \hat{\omega} = \omega_2\,(e)\,, \tag{23}$$

both independent of $\bar{\sigma}$, recovered by setting $\phi_1\,(\bar{\sigma}) = 1$. Then

$$F = -\,f\,(e) + F_2\,(e)\,\bar{\sigma},\quad \hat{E} = \frac{1}{F_2\,(e)\,\hat{\psi}\,(\bar{\sigma},\,e)}\,, \tag{24}$$

requiring F linear in $\bar{\sigma}$ but $\hat{E}$ unrestricted.

If the restricted forms (13) and (14), with separable $\hat{E}$, are applied to (17) and (18), then

$$F = -\,f(e) + \frac{\bar{\sigma}\,\hat{\psi}^*\,(\bar{\sigma},\,e)}{\omega_1\,(\bar{\sigma})\,E_2\,(e)}\,, \tag{25}$$

is restricted, and

$$\frac{3}{2}\,\hat{\phi}^* = (1-e)^2\,\omega_1\,(\bar{\sigma})\,E_2\,(e)\,,\qquad \hat{\psi}^* = \frac{\omega_1\,(\bar{\sigma})\,E_2\,(e)\,[F\,(\bar{\sigma},\,e) + f\,(e)]}{\bar{\sigma}}\,. \tag{26}$$

Either $\omega_1\,(\bar{\sigma})$, hence $\hat{E}$, or $F\,(\bar{\sigma},\,e) + f\,(e)$ must be order $\bar{\sigma}$ (or smaller) as $\bar{\sigma} \to 0$ for bounded $\hat{\psi}^*$. The special case

$$\hat{\omega}^* = \omega_2\,(e)\,,\quad \hat{E} = E_2\,(e) \tag{27}$$

is recovered by setting $\omega_1\,(\bar{\sigma}) = 1$, and $F\,(\bar{\sigma},\,e) + f\,(e)$ must be order $\bar{\sigma}$ (or smaller) as $\bar{\sigma} \to 0$, but $F\,(\bar{\sigma},\,e)$ is otherwise unrestricted. Given that the response $\dot{e} = F(\bar{\sigma}, e)$ to constant $\bar{\sigma}$ should be closely matched, the reduced model (13) and (14) is preferable to (11) and (12), and the associated separable from $(13)_2$ of $\hat{E}$ could be an adequate approximation since illustrations have shown that variation of $\hat{E}$ with e has little effect on the constant displacement rate response (Spring and Morland 1981).

Finally, consider the relaxation on unloading governed by (10), which is a given feature of the viscoelastic solid model.
Since

$$t - t_1 = \int_e^{e_1} \frac{de'}{f(e')}, \tag{28}$$

then if $e \to e_\infty$ as $t \to \infty$, and if

$$0 \leq e_\infty \leq e_1 \tag{29}$$

so that the final strain is not a stretch and not a larger compression than at time t_1+

$$\int_{e_\infty}^{e_1} \frac{de'}{f(e')} = +\infty. \tag{30}$$

Now $f^{-1}(e)$ is integrable near e_1 to obtain bounded t by (28) for some initial relaxation (the case $f \equiv 0$, implying no relaxation, is excluded), so $f^{-1}(e)$ must be nonintegrable near e_∞. But these results must apply for any initial strain $e_1 > 0$, so in particular for $e_\infty = 0$. Thus $f^{-1}(e)$ in nonnegative and non-integrable as $e \to 0+$, and has positive infinite integral. Hence

$$f(e) \sim f_o e^n \ (n \geq 1) \text{ as } e \to 0. \tag{31}$$

A simple example is

$$f(e) = ke, \qquad e = e_1 \exp[-k(t-t_1)] \text{ for } t \geq t_1. \tag{32}$$